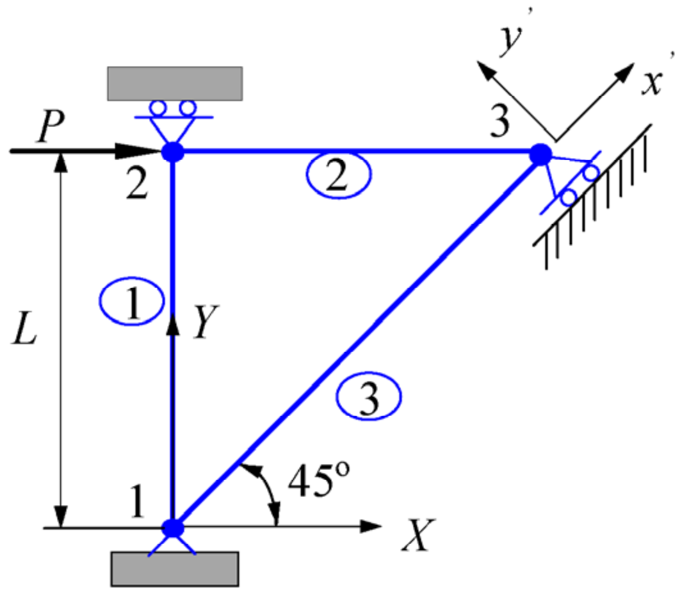
Example 2.4 (Multipoint Constraint)



For the plane truss shown above,

$$P = 1000 \text{ kN}, \quad L = 1m, \quad E = 210 GPa,$$

$$A = 6.0 \times 10^{-4} m^2 \quad \text{for elements 1 and 2,}$$

$$A = 6\sqrt{2} \times 10^{-4} m^2 \quad \text{for element 3.}$$

Determine the displacements and reaction forces.

Solution:

We have an inclined roller at node 3, which needs special attention in the FE solution. We first assemble the global FE equation for the truss.

Element 1:

$$\theta = 90^o, \quad l = 0, \quad m = 1$$

$$\mathbf{k}_1 = \frac{(210\times 10^9)(6.0\times 10^{-4})}{1}\begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_2 & v_2 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \end{array} (\mathrm{N/m})$$

Element 2:

$$\theta = 0^o, \quad l = 1, \quad m = 0$$

$$\mathbf{k}_2 = \frac{(210\times 10^9)(6.0\times 10^{-4})}{1}\begin{array}{c} \begin{array}{cccc} u_2 & v_2 & u_3 & v_3 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array} (\mathrm{N/m})$$

Element 3:

$$\theta = 45^o, \quad l = \frac{1}{\sqrt{2}}, \quad m = \frac{1}{\sqrt{2}}$$

$$\mathbf{k}_3 = \frac{(210\times 10^9)(6\sqrt{2}\times 10^{-4})}{\sqrt{2}}\begin{array}{c} \begin{array}{cccc} u_1 & v_1 & u_3 & v_3 \end{array} \\ \begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix} \end{array}$$

$$(\mathrm{N/m})$$

The global FE equation is,

$$1260\times10^{5}\begin{bmatrix} 0.5 & 0.5 & 0 & 0 & -0.5 & -0.5 \\ & 1.5 & 0 & -1 & -0.5 & -0.5 \\ & & 1 & 0 & -1 & 0 \\ & & & 1 & 0 & 0 \\ & & & & 1.5 & 0.5 \\ Sym. & & & & & 0.5 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}=\begin{Bmatrix} F_{1X} \\ F_{1Y} \\ F_{2X} \\ F_{2Y} \\ F_{3X} \\ F_{3Y} \end{Bmatrix}$$

Load and boundary conditions (BC):

$$u_1 = v_1 = v_2 = 0, \text{ and } v_3' = 0,$$
$$F_{2X} = P, \;\; F_{3x'} = 0.$$

From the transformation relation and the BC, we have

$$v_3' = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}\begin{Bmatrix} u_3 \\ v_3 \end{Bmatrix} = \frac{\sqrt{2}}{2}(-u_3 + v_3) = 0,$$

that is,

$$u_3 - v_3 = 0$$

This is a *multipoint constraint* (MPC).

Similarly, we have a relation for the force at node 3,

$$F_{3x'} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}\begin{Bmatrix} F_{3X} \\ F_{3Y} \end{Bmatrix} = \frac{\sqrt{2}}{2}(F_{3X} + F_{3Y}) = 0,$$

that is,

$$F_{3X} + F_{3Y} = 0$$

Applying the load and BC's in the structure FE equation by 'deleting' 1st, 2nd and 4th rows and columns, we have

$$1260\times10^5\begin{bmatrix}1 & -1 & 0\\ -1 & 1.5 & 0.5\\ 0 & 0.5 & 0.5\end{bmatrix}\begin{Bmatrix}u_2\\ u_3\\ v_3\end{Bmatrix}=\begin{Bmatrix}P\\ F_{3X}\\ F_{3Y}\end{Bmatrix}$$

Further, from the MPC and the force relation at node 3, the equation becomes,

$$1260\times10^5\begin{bmatrix}1 & -1 & 0\\ -1 & 1.5 & 0.5\\ 0 & 0.5 & 0.5\end{bmatrix}\begin{Bmatrix}u_2\\ u_3\\ u_3\end{Bmatrix}=\begin{Bmatrix}P\\ F_{3X}\\ -F_{3X}\end{Bmatrix}$$

which is

$$1260\times10^5\begin{bmatrix}1 & -1\\ -1 & 2\\ 0 & 1\end{bmatrix}\begin{Bmatrix}u_2\\ u_3\end{Bmatrix}=\begin{Bmatrix}P\\ F_{3X}\\ -F_{3X}\end{Bmatrix}$$

The 3rd equation yields,

$$F_{3X} = -1260\times10^5 u_3$$

Substituting this into the 2nd equation and rearranging, we have

$$1260\times10^5\begin{bmatrix}1 & -1\\ -1 & 3\end{bmatrix}\begin{Bmatrix}u_2\\ u_3\end{Bmatrix}=\begin{Bmatrix}P\\ 0\end{Bmatrix}$$

Solving this, we obtain the displacements,

$$\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \frac{1}{2520 \times 10^5} \begin{Bmatrix} 3P \\ P \end{Bmatrix} = \begin{Bmatrix} 0.01191 \\ 0.003968 \end{Bmatrix} \quad \text{(m)}$$

From the global FE equation, we can calculate the reaction forces,

$$\begin{Bmatrix} F_{1X} \\ F_{1Y} \\ F_{2Y} \\ F_{3X} \\ F_{3Y} \end{Bmatrix} = 1260 \times 10^5 \begin{bmatrix} 0 & -0.5 & -0.5 \\ 0 & -0.5 & -0.5 \\ 0 & 0 & 0 \\ -1 & 1.5 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} -500 \\ -500 \\ 0.0 \\ -500 \\ 500 \end{Bmatrix} \text{(kN)}$$

Check the results!

A general *multipoint constraint* (MPC) can be described as,

$$\sum_j A_j u_j = 0$$

where A_j's are constants and u_j's are nodal displacement components. In the FE software, such as *MSC/NASTRAN*, users only need to specify this relation to the software. The software will take care of the solution.

Penalty Approach for Handling BC's and MPC's