بسم الله الرحمن الرحيم

King Abdulaziz University
Engineering College
Department of Production and Mechanical System Design

MENG 470 Mechanical Vibrations

Final Exam
Closed-book Exam
Wednesday: 24/11/1425 H
Time Allowed: 120 mins

Name:	ID No.:

Question 1		25
Question 2		40
Question 3		35
TOTAL		100

قال الله تعالى : (تلك الدار الآخرة نجعلها للذين لا يريدون علوا في الأرض ولا فسادا والعاقبة للمتقين)

Instructions

1. This is a closed book and closed notes Opportunity to Shine
2. Show all work for partial credit.
3. Assemble your work for each problem in logical order.
4. Justify your conclusion. I cannot read minds.

Q1. Indicate whether each of the following statements is **true** or **false**:

1. The amplitude of an undamped system will not change with time.
2. A system vibrating in air can be considered a damped system.
3. The equation of motion of a single degree of freedom system will be the same whether the mass moves in a horizontal plane or an inclined plane.
4. When a mass vibrates in a vertical direction, its weight can always be ignored in deriving the equation of motion.
5. The principle of conservation of energy can be used to derive the equation of motion of both damped and undamped systems.
6. The damped frequency can in some cases be larger than the undamped natural frequency of the system.
7. The damped frequency can be zero in some cases.
8. The natural frequency of vibration of a torsional system is given by $\sqrt{k_T / J}$, where k_T and J denote the torsional spring constant and the polar mass moment of inertia, respectively.
9. The undamped natural frequency of a system is given by $\sqrt{g / \delta_{st}}$ where δ_{st} is the static deflection of the mass.
10. For an undamped system, the velocity leads the displacement by $\pi / 2$.
11. The motion diminishes to zero in both underdamped and overdamped cases.
12. The logarithmic decrement can be used to find the damping ratio.
13. In torsional vibration, the displacement is measured in terms of linear coordinate
14. The phase angle of the response depends on the system parameter *m, c, k,* and ω .
15. During beating, the amplitude of the response builds up and then diminishes in a regular pattern.
16. The *Q*-factor can be used to estimate the damping in a system.
17. The amplitude ratio attains its maximum value at resonance in the case of viscous damping.
18. Damping reduces the amplitude ratio for all values of the forcing frequency.
19. The unbalance in a rotating machine causes vibration.
20. The normal modes can also be called principal modes.
21. The generalized coordinates are linearly dependent.
22. Principal coordinates can be considered as generalized coordinates.
23. The vibration of a system depends on the coordinate system.
24. The nature of coupling depends in the coordinate system.
25. The magnification factor is the ratio of maximum amplitude and static deflection.
26. The response will be harmonic if excitation is harmonic.

27. The principal (or modal) coordinates avoid both static and dynamic coupling.
28. The use of principal (or modal) coordinates can NOT be used to find the response of the system.
29. The mass, stiffness, and damping matrices of a two degree of freedom system are always NOT symmetric.
30. The characteristics of a two degree of freedom system are used in the design of dynamic vibration absorber.
31. A semidefmite system can NOT have nonzero natural frequencies.
32. During free vibration, different degrees of freedom oscillate with different amplitudes.
33. The modal eigenvectors of a system are the physical not-normalized modes of vibration.
34. The vibration of a system under external forces is called damped vibration.
35. When a two degree of freedom system is subjected to a harmonic force, the system vibrates at the frequency of applied force.
36. When the forcing frequency is equal to one of the natural frequencies of the system, a phenomenon known as *beating* occurs.
37. For a underdamped multidegree of freedom system, all the eigenvalues can be complex.
38. The amplitudes and phase angles are determined from the boundary conditions of the system.
39. A definite system has at least one rigid body motion.
40. The elastic coupling is also known as dynamic coupling while the inertia coupling is also known as static coupling.
41. The equations of motion of a system will be coupled when principal (or principle) coordinates are used.
42. The vibration of a system under initial conditions only is called forced vibration.
43. The number of degrees of freedom of a vibrating system depends only on number of masses.
44. The equations of motion of a two degree of freedom system are in general coupled.
45. The stiffness matrix of a system is always symmetric and positive definite.
46. For a multidegree of freedom system, one equation of motion can be written for each degree of freedom.
47. Lagrange's equation cannot be used to derive the equations of motion of a multidegree of freedom system.
48. The mass, stiffness, and damping matrices of a multidegree of freedom are always symmetric.
49. A multidegree of freedom system can have six of the natural frequencies equal to zero
50. The mass matrix of a system is always symmetric and positive definite.

Answers:

	True	False
1.	●	
2.	●	
3.	●	
4.	●	
5.		●
6.		●
7.	●	
8.	●	
9.	●	
10.	●	
11.	●	
12.	●	
13.		●
14.	●	
15.	●	
16.	●	
17.	●	
18.		●
19.	●	
20.	●	
21.	●	
22.	●	
23.	●	
24.	●	
25.	●	
26.	●	
27.	●	
28.		●
29.		●
30.	●	
31.		●
32.	●	
33.		●
34.		●
35.		●
36.		●
37.		●
38.	●	
39.		●
40.		●
41.		●
42.		●
43.		●
44.	●	
45.		●
46.	●	
47.		●
48.	●	
49.		●
50.	●	

Q2. Consider the system shown in Figure 1 where m_1 = 30 kg, m_2 = 2 kg, k = 15 N/m, l=2m, $f(t)$ = 10 sin(5t) N.

(a) What's the degree of the system?

(b) Write the equation of motion of the system in matrix form.

(c) Is the system statically or dynamically coupled or both.

(d) Find the natural frequencies and corresponding mode shapes.

(e) Calculate the normalized eigenvectors of the system.

(f) Write down the matrix form.

(g) Decouple the coupled equations using modal transformation.

(h) Recover the physical degrees of freedom from the modal degree of freedom.

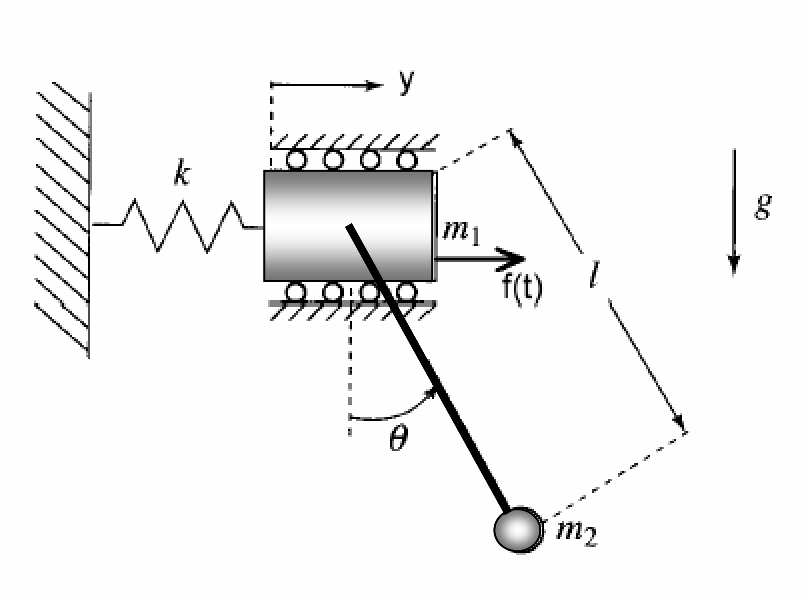

Figure 1

Answer:

$$T = \frac{1}{2} m_1 \dot{y}^2 + \frac{1}{2} m_2 (\dot{y} + \ell \dot{\theta})^2$$

$$U = \frac{1}{2} k y^2 + m_2 g \ell (1 - \cos\theta) = \frac{1}{2} k y^2 + m_2 g \ell \frac{\theta^2}{2}$$

$$\frac{\partial T}{\partial \dot{y}} = m_1 \dot{y} + m_2 \dot{y} + m_2 \ell \dot{\theta}$$

$$\frac{\partial T}{\partial \dot{\theta}} = m_2 \ell \dot{y} + m_2 \ell^2 \dot{\theta}$$

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{y}}\right] = m_1 \ddot{y} + m_2 \ddot{y} + m_2 \ell \ddot{\theta}$$

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{\theta}}\right] = m_2 \ell \ddot{y} + m_2 \ell^2 \ddot{\theta}$$

$$\frac{\partial U}{\partial \dot{y}} = k_1 y$$

$$\frac{\partial U}{\partial \dot{\theta}} = m_2 g \ell \theta$$

$$\begin{bmatrix} (m_1 + m_2) & m_2 \ell \\ m_2 \ell & m_2 \ell^2 \end{bmatrix} \begin{Bmatrix} \ddot{y} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} k_1 & 0 \\ 0 & m_2 g \ell \end{bmatrix} \begin{Bmatrix} y \\ \theta \end{Bmatrix} = \begin{Bmatrix} f(t) \\ 0 \end{Bmatrix}$$

Substituting numbers ($m_1 = 30$ kg, $m_2 = 2$ kg, $k_1 = 15$ N/m, $\ell = 2$m, and $g = 9.81\text{m/s}^2$, a $f(t) = 10\sin(5t)$N) gives:

$$\begin{bmatrix} 32 & 4 \\ 4 & 8 \end{bmatrix} \begin{Bmatrix} \ddot{y} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} 15 & 0 \\ 0 & 39.24 \end{bmatrix} \begin{Bmatrix} y \\ \theta \end{Bmatrix} = \begin{Bmatrix} 10\sin(5t) \\ 0 \end{Bmatrix}$$

where the inertia coupling is clear. The goal is to find the solution to this coupled set of differential equations.

Solving the eigenvalue problem gives (this was done on the last assignment):

$$\text{first mode:}\quad \omega_1^2 = .4657\ (\text{rad/s})^2 \quad \{X\}_1 = \begin{Bmatrix} Y \\ \Theta \end{Bmatrix}_1 = \begin{Bmatrix} 19.065 \\ 1.000 \end{Bmatrix}$$

$$\text{second mode:}\quad \omega_2^2 = 5.266\ (\text{rad/s})^2 \quad \{X\}_2 = \begin{Bmatrix} Y \\ \Theta \end{Bmatrix}_2 = \begin{Bmatrix} -0.137 \\ 1.000 \end{Bmatrix}$$

The first mode shape is normalized as follows:

a. Calculate

$$\{X\}_1^T [M] \{X\}_1 = \begin{Bmatrix} 19.065 & 1.000 \end{Bmatrix} \begin{bmatrix} 32 & 4 \\ 4 & 8 \end{bmatrix} \begin{Bmatrix} 19.065 \\ 1.000 \end{Bmatrix} = 1.179 \times 10^4$$

b. Scale the eigenvector by dividing each element by the square root of the number in part

$$\{X\}_1 = \left\{ \begin{array}{c} Y \\ \Theta \end{array} \right\}_1 = \left\{ \begin{array}{c} 19.065/\sqrt{1.179 \times 10^4} \\ 1.000/\sqrt{1.179 \times 10^4} \end{array} \right\} = \left\{ \begin{array}{c} 0.17557 \\ 0.00921 \end{array} \right\}$$

c. Now,

$$\{X\}_1^T [M] \{X\}_1 = \left\{ \begin{array}{cc} 0.17557 & 0.00921 \end{array} \right\} \left[\begin{array}{cc} 32 & 4 \\ 4 & 8 \end{array} \right] \left\{ \begin{array}{c} 0.17557 \\ 0.00921 \end{array} \right\} = 1.00$$

The second mode shape yields:

$$\{X\}_2^T [M] \{X\}_2 = \left\{ \begin{array}{cc} -0.137 & 1.000 \end{array} \right\} \left[\begin{array}{cc} 32 & 4 \\ 4 & 8 \end{array} \right] \left\{ \begin{array}{c} -0.137 \\ 1.000 \end{array} \right\} = 7.5048$$

and the scaled mode is:

$$\{X\}_2 = \left\{ \begin{array}{c} Y \\ \Theta \end{array} \right\}_2 = \left\{ \begin{array}{c} -0.137/\sqrt{7.5048} \\ 1.000/\sqrt{7.5048} \end{array} \right\} = \left\{ \begin{array}{c} -0.0501 \\ 0.3650 \end{array} \right\}$$

The modal matrix is then:

$$[U] = [\{X\}_1 \{X\}_2] = \left[\begin{array}{cc} 0.17557 & -0.0501 \\ 0.00921 & 0.3650 \end{array} \right]$$

The vector of degrees-of-freedom can be expanded in terms of the eigenvectors as:

$$\left\{ \begin{array}{c} y(t) \\ \theta(t) \end{array} \right\} = c_1(t) \left\{ \begin{array}{c} Y \\ \Theta \end{array} \right\}_1 + c_2(t) \left\{ \begin{array}{c} Y \\ \Theta \end{array} \right\}_2$$

where the $c_i(t)$ variables represent the fraction of each mode contributing to the values of the degrees-of-freedom, $y(t)$ and $\theta(t)$, at any time t.
This can also be expressed as:

$$\left\{ \begin{array}{c} y(t) \\ \theta(t) \end{array} \right\} = [U] \left\{ \begin{array}{c} c_1(t) \\ c_2(t) \end{array} \right\}$$

Substituting this into the equations-of-motion yields:

$$[M] [U] \{\ddot{c}\} + [K] [U] \{c\} = \{f\}$$

Premultiply this by the transpose of the modal matrix to give:

$$[U]^T [M] [U] \{\ddot{c}\} + [U]^T [K] [U] \{c\} = [U]^T \{f\}$$

where the normalization of the eigenvectors causes:

$$[U]^T [M] [U] = [I]$$

where $[I]$ is the identity matrix and:

$$[U]^T [K] [U] = \left[\text{diag}(\omega_i^2)\right]$$

where $[\text{diag}(\omega_i^2)]$ is a diagonal matrix with the squared natural frequencies as elements.

For the particular example here:

$$[U]^T [M] [U] = \begin{bmatrix} 0.17557 & 0.00921 \\ -0.0501 & 0.3650 \end{bmatrix} \begin{bmatrix} 32 & 4 \\ 4 & 8 \end{bmatrix} \begin{bmatrix} 0.17557 & -0.0501 \\ 0.00921 & 0.3650 \end{bmatrix} = \begin{bmatrix} 1.00 & 0.00 \\ 0.00 & 1.00 \end{bmatrix}$$

$$[U]^T [K] [U] = \begin{bmatrix} 0.17557 & 0.00921 \\ -0.0501 & 0.3650 \end{bmatrix} \begin{bmatrix} 15 & 0 \\ 0 & 39.24 \end{bmatrix} \begin{bmatrix} 0.17557 & -0.0501 \\ 0.00921 & 0.3650 \end{bmatrix} = \begin{bmatrix} .4657 & 0.00 \\ 0.00 & 5.266 \end{bmatrix}$$

and:

$$[U]^T \{f\} = \begin{bmatrix} 0.17557 & 0.00921 \\ -0.0501 & 0.3650 \end{bmatrix} \begin{Bmatrix} 10\sin(5t) \\ 0 \end{Bmatrix} = \begin{Bmatrix} 1.7557\sin(5t) \\ -0.501\sin(5t) \end{Bmatrix}$$

The equations-of-motion then become:

$$\begin{bmatrix} 1.00 & 0.00 \\ 0.00 & 1.00 \end{bmatrix} \begin{Bmatrix} \ddot{c}_1 \\ \ddot{c}_2 \end{Bmatrix} + \begin{bmatrix} .4657 & 0.00 \\ 0.00 & 5.266 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \begin{Bmatrix} 1.7557\sin(5t) \\ -0.501\sin(5t) \end{Bmatrix}$$

The system of equations has now been decoupled and can be written as two, independent equations that can be solved using the methods applied to single degree-of-freedom systems. That is:

$$\ddot{c}_1 + .4657\ c_1 = 1.7557\sin(5t)$$

and

$$\ddot{c}_2 + 5.266\ c_2 = -0.501\sin(5t)$$

The physical degrees-of-freedom can be recovered at any time from:

$$\begin{Bmatrix} y(t) \\ \theta(t) \end{Bmatrix} = [U] \begin{Bmatrix} c_1(t) \\ c_2(t) \end{Bmatrix} = \begin{bmatrix} 0.17557 & -0.0501 \\ 0.00921 & 0.3650 \end{bmatrix} \begin{Bmatrix} c_1(t) \\ c_2(t) \end{Bmatrix}$$

Q3. Consider a cable shown in Figure 2 that has one end fixed and the other end free to slide along a smooth vertical guide. The free end cannot support a transverse force so that we have:

$$\frac{\partial \omega(L,t)}{\partial x} = 0$$

The cable length L=100m is made out of steel with a uniform density ρ=7.8X10^{3} kg/m^{3}, and constant cross sectional area A=7.854X10^{-5} m^{2}; and it is under tension of T=10,000 N.

Calculate the natural frequencies and mode shape of the cable. Plot the first four made shapes (Normalized the mode shapes so that its maximum amplitude is one).

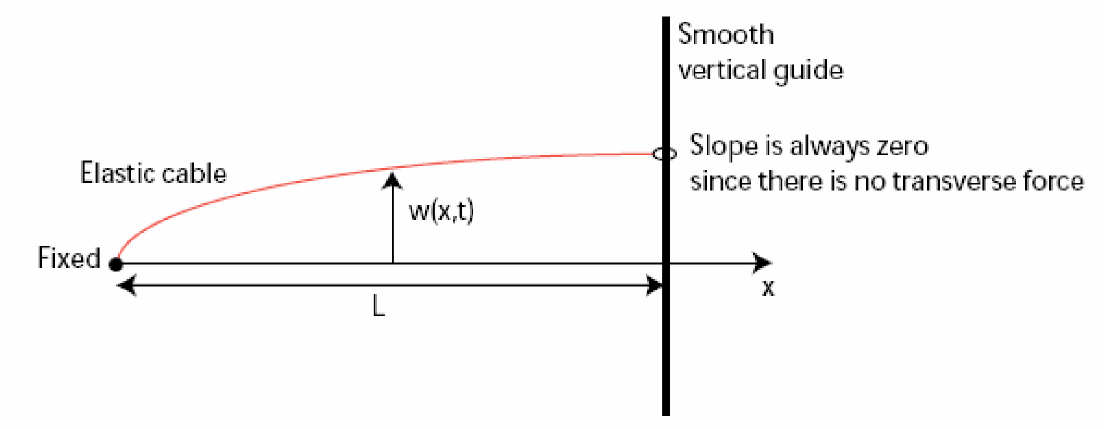

Figure 2

Answer:

Calculate the natural frequencies and mode shapes of the cable. Plot the ***first four mode shapes*** (Normalize the mode shape so that its maximum amplitude is one).

The equation of motion for the cable is

$$\frac{\partial^2 w(x,t)}{\partial^2 t} = c^2 \frac{\partial^2 w(x,t)}{\partial^2 x}$$

where

$$c^2 = \frac{\tau}{\rho A} = 1.632354 \times 10^4$$

The wave speed is $c = 127.76$ m/sec.

Using the method of separation of variables, we have

$$\ddot{T}(t) + \omega^2 T(t) = 0 \qquad (1)$$

and

$$X''(x) + \frac{\omega^2}{c^2} X(x) = 0 \qquad (2)$$

with boundary conditions $X(0) = 0$ and $X'(L) = 0$.

The mode shape function $X(x)$ is a solution of equation 2 given by

$$X(x) = A\cos(\frac{\omega}{c}x) + B\sin(\frac{\omega}{c}x)$$

$$X'(x) = -A\frac{\omega}{c}\sin(\frac{\omega}{c}x) + B\frac{\omega}{c}\cos(\frac{\omega}{c}x)$$

Applying the two boundary conditions, we have

$$X(0) = 1A + 0B = 0 \quad , \quad -\frac{\omega}{c}\sin(\frac{\omega}{c}L)A + \frac{\omega}{c}\cos(\frac{\omega}{c}L)B = 0$$

Or re-writing them in a matrix equation, we have

$$\begin{bmatrix} 1 & 0 \\ -\frac{\omega}{c}\sin(\frac{\omega}{c}L) & \frac{\omega}{c}\cos(\frac{\omega}{c}L) \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad (3)$$

For nontrivial solutions in A and B, the matrix $\begin{bmatrix} 1 & 0 \\ -\frac{\omega}{c}\sin(\frac{\omega}{c}L) & \frac{\omega}{c}\cos(\frac{\omega}{c}L) \end{bmatrix}$ must be SINGULAR; thus we obtain the characteristic equation of the cable system as

$$det\begin{bmatrix} 1 & 0 \\ -\frac{\omega}{c}\sin(\frac{\omega}{c}L) & \frac{\omega}{c}\cos(\frac{\omega}{c}L) \end{bmatrix} = 0 \Longrightarrow \frac{\omega}{c}\cos(\frac{\omega}{c}L) = 0$$

The natural frequencies are given by

$$\frac{\omega}{c} = 0 \quad , \quad \cos(\frac{\omega}{c}L) = 0$$

Note that the natural frequency $\omega = 0$ leads to a trivial solution of $X(x) = 0$.

$$\cos(\frac{\omega}{c}L) = 0 \Longrightarrow \frac{\omega}{c}L = \frac{2k+1}{2}\pi$$

Or

$$\omega_k = \frac{(2k+1)\pi}{2}\frac{c}{L} \ \text{ rad/sec} \quad (k = 0, 1, 2, 3, \cdots)$$

Solving for the mode shape from equation (3), we have

$$A = 0 \quad , \quad B = 1$$

Thus, the k^{th} mode shape is given by

$$X_k(x) = \sin(\frac{(2k+1)\pi}{2}\frac{x}{L})$$

The first four mode shapes and natural frequencies are

(a) For $k = 0$

$$\omega_1 = \frac{\pi}{2}\frac{127.76}{100} = 2.0069 \ \text{ rad/sec}$$

$$X_1(x) = \sin(\frac{\pi}{2}\frac{x}{L})$$

(b) For $k = 1$

$$\omega_2 = \frac{3\pi}{2}\frac{127.76}{100} = 6.0207 \ \text{ rad/sec}$$

$$X_2(x) = \sin(\frac{3\pi}{2}\frac{x}{L})$$

Q.4 Consider the system shown in Figure 3 and determine the following:

a) The degree of freedom.

b) The kinetic energy of the system in terms of $\dot{x}$.

c) The potential energy of the system in terms of *x*.

d) The equation (or equations) of motion.

e) The natural frequency (or frequencies).

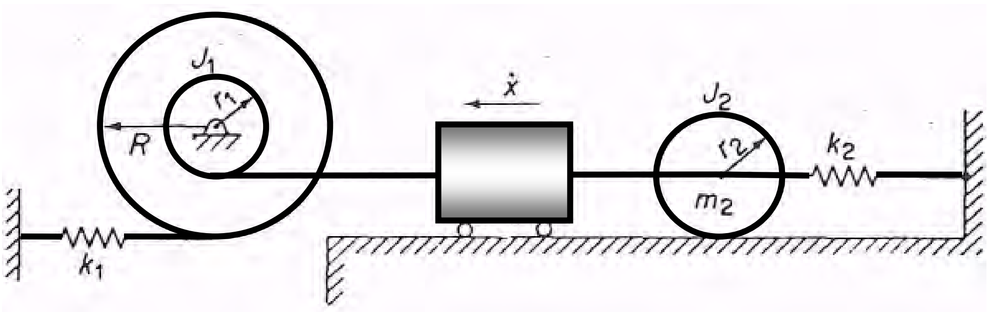

Figure 3

Answer:

The system is a 1-DOF system.

$$T = \frac{1}{2}J_1\dot{\theta}_1^2 + \frac{1}{2}(m_0 + m_2)\dot{x}^2 + \frac{1}{2}J_2\dot{\theta}_2^2$$

$$\dot{\theta}_1 = \frac{\dot{x}}{r_1} \quad \text{and} \quad \dot{\theta}_2 = \frac{\dot{x}}{r_2}$$

$$T = \frac{1}{2}\left[\frac{J_1}{r_1^2} + (m_0 + m_2) + \frac{J_2}{r_2^2}\right]\dot{x}^2 = \frac{1}{2}m_{eff}\,\dot{x}^2$$

$$U = \frac{1}{2}k_1(R\theta_1)^2 + \frac{1}{2}k_2(r_2\theta_2)^2 = \frac{1}{2}\left[k_1\left(\frac{R}{r_1}\right)^2 + k_2\right]x^2 = \frac{1}{2}k_{eff}\,x^2$$

The equation of motion can now be written as:

$$m_{eff}\,\ddot{x} + k_{eff}\,x = 0$$

$$\left[\frac{J_1}{r_1^2} + (m_0 + m_2) + \frac{J_2}{r_2^2}\right]\ddot{x} + \left[k_1\left(\frac{R}{r_1}\right)^2 + k_2\right]x = 0$$

The natural frequency is:

$$\left[\frac{J_1}{r_1^2} + (m_0 + m_2) + \frac{J_2}{r_2^2}\right]\ddot{x} + \left[k_1\left(\frac{R}{r_1}\right)^2 + k_2\right]x = 0$$

$$\omega_n = \sqrt{\frac{k_{eff}}{m_{ff}}}$$

$$\omega_n = \sqrt{\frac{k_1\left(\frac{R}{r_1}\right)^2 + k_2}{\frac{J_1}{r_1^2} + (m_0 + m_2) + \frac{J_2}{r_2^2}}}$$

Q5. The system shown in Figure 4 has the following parameters:

$m = 1$ kg, $I_G = 2$ kg.m^2, $k_1 = k_2 = 100$ N/m, $r = 2$ m, $M_2(t) = 100\cos\omega t$

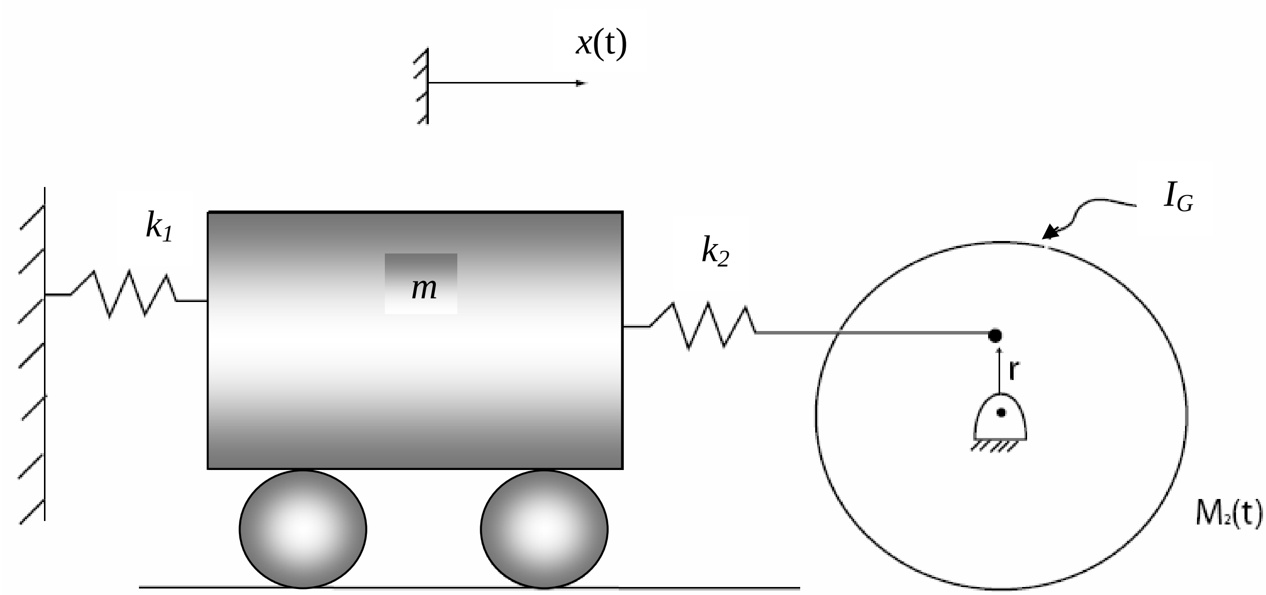

Figure 3

Figure1: A two degree of freedom system with translation and rotation.

a) Derive the equations of motion.

b) Find the natural frequencies for the system ω_1 and ω_2.

c) Find the "mass normalized" eigenvectors (U).

d) Find $U^T MU$ and $U^T KU$.

e) Decouple the equations of motion into modal coordinates and find the transient and steady state solution, or modal displacements, for each modal coordinate (η_1 and η_2).

Use the following initial conditions: $x(0) = 0$, $\dot{x}(0) = 4$, $\theta(0) = 0$, $\dot{\theta}(0) = 0$.

f) Use the solution in modal coordinates to write the physical displacement of m and physical rotation of IG, or the position vector. $X = [x \ \ \theta]^T$.

1. In matrix notation EOM is:-

$$\begin{bmatrix} m & 0 \\ 0 & I_g \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} K_1+K_2 & -K_2 r \\ -K_2 r & K_2 r^2 \end{bmatrix} \begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ M_2(t) \end{Bmatrix}$$

Plugging in the values.

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} 200 & -200 \\ -200 & 400 \end{bmatrix} \begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 100 \cos \Omega t \end{Bmatrix}$$

2. (a) Natural frequencies of the system :-

$\omega_1 = 7.6531$ rad/s

& $\omega_2 = 18.47$ rad/s.

(b) Mass Normalized Eigen Vectors:-

Now $V = \begin{bmatrix} 1 & 1 \\ .7071 & -.7071 \end{bmatrix}$

alpha 1 = $\sqrt{\dfrac{1}{(m \times V(1,1)^2 + I_g \times (V_{2,1})^2)}}$

Substituting the values $V_{1,1} = 1$ & $V_{2,1} = 0.7071$

alpha 1 = 1

Similarly

alpha 2 = $\sqrt{\dfrac{1}{(m \times V(1,2)^2 + I_g \times V(2,2)^2)}}$

Substituting $V_{1,2} = 1$ & $V_{2,2} = -.7071$ we have

alpha2 = 1

∴ <u>Mass Normalized Eigen Vectors are</u> :-

$$U = \begin{bmatrix} -.7071 & -.7071 \\ -0.5 & 0.5 \end{bmatrix}$$

(c) $U^T M U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ⇐ Identity Matrix

& $U^T K U = \begin{bmatrix} 58.57 & 0 \\ 0 & 341.421 \end{bmatrix}$ ⇐ Diagonal matrix

3. The equations of motion in in modal co-ordinates are

a) Transfer forces to modal coordinates:-

$[F_m] = [U]^T \{F\}$ let $\Omega = \frac{1}{2}(7.65 + 18.47) \simeq 13$ rad/s

$$= \begin{bmatrix} -.7071 & -0.5 \\ -.7071 & +0.5 \end{bmatrix} \begin{Bmatrix} 0 \\ 100\cos(13t) \end{Bmatrix}$$

$$\Rightarrow [F_m] = \begin{Bmatrix} -50\cos(13t) \\ 50\cos(13t) \end{Bmatrix}.$$

b) Transfer Initial conditions to Modal Coordinates:-

$$\begin{Bmatrix} x_m \\ \theta_m \end{Bmatrix} = \begin{bmatrix} -.7071 & -0.5 \\ -.7071 & +0.5 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}.$$

Similarly

$$\begin{Bmatrix} \dot{x}_m \\ \dot{\theta}_m \end{Bmatrix} = \begin{bmatrix} -.7071 & -0.5 \\ -.7071 & +0.5 \end{bmatrix} \begin{Bmatrix} 4 \\ 0 \end{Bmatrix} = \begin{Bmatrix} -2.82 \\ -2.82 \end{Bmatrix}$$

Writing EOM in modal co-ordinate system

$$(1)\ddot{\eta}_1 + (58.57)\eta_1 = -50\cos(13t) \quad \text{—(i)}$$

& $$(1)\ddot{\eta}_2 + (341.421)\eta_2 = 50\cos(13t) \quad \text{—(ii)}$$

Considering Each EOM separately.

(A) $\ddot{\eta}_1 + 58.57\eta_1 = -50\cos(13t)$ — (iii)

I C's $\eta_1(0) = 0$ & $\dot{\eta}_1(0) = -2.82$

This now becomes SDOF, undamped, forced vibration problem.

$\eta_1 = \eta_{1h} + \eta_{1p}$

For homogeneous solⁿ:

Now let $\eta_1 = Ae^{\lambda t}$ & $\dot{\eta}_2 = \lambda Ae^{\lambda t}$ & $\ddot{\eta}_2 = \lambda^2 Ae^{\lambda t}$

Substitute into (iii)

$(\lambda^2 + 58.57)Ae^{\lambda t} = 0$

$\Rightarrow (\lambda^2 + 58.57) = 0$ {NON TRIVIAL SOLUTION}.

$\Rightarrow \lambda = \pm 7.65i$

$\Rightarrow \eta_{1,h} = Ae^{-7.65it} + Be^{7.65it}$ ——— (iv)

forcing frequency.

Let $\eta_p = De^{i(13)t}$

$\Rightarrow \dot{\eta}_p = 13i\,De^{i13t}$

& $\ddot{\eta}_p = -(13)^2 De^{i13t}$

Substituting in (iii) we have:-

$-(13)^2 De^{i13t} + (58.57)De^{i(13)t} = -50e^{i13t}$

$\Rightarrow [-(13)^2 + (58.57)]D = -50$

$\Rightarrow D = \frac{-50}{-110.43} = .45$

$$\Rightarrow \eta_1 = Ae^{-7.65it} + Be^{7.65it} + (.45)e^{i13t}$$

$$\eta_1 = Ae^{-7.65it} + Be^{7.65it} + .45e^{i13t}$$

<u>Applying the initial conditions</u>

@ $t=0$ $\eta_1 = 0$ and $\dot{\eta}_1 = -2.82$

$\therefore$ $0 = A + B + 0.45$

$\Rightarrow A + B = -0.45$ — (v)

$$\dot{\eta}_1 = -7.65i\, Ae^{-7.65it} + 7.65i\, Be^{7.65it} + (5.85)ie^{i13t}$$

@ $t=0$

$$-2.82 = -7.65i\, A + 7.65i\, B + 5.85\, i$$

$$\frac{-5.85i - 2.82}{7.65i} = -A + B.$$

$\Rightarrow$ $-A + B = -0.764 + .368i$ — (vi)

from (v) & (vi)

$2B = -1.214 + .368i$

$B = .607 + .184i$

& $A = -1.057 - .184i$

$$\Rightarrow \eta_1 = (-1.05 - .184i)e^{-7.65it} + (.607 + .184i)e^{7.65it} - .45e^{i13t}$$

(B) $\ddot{\eta}_2 + (341.421)\eta_2 = 50\cos(13t)$

$\eta_2(0) = 0$ & $\dot{\eta}_2(0) = -2.82$

$\eta_2 = \eta_{2h} + \eta_{2p}$

for homogeneous sol^n

let $\eta_2 = Ae^{\lambda t} \Rightarrow \ddot{\eta}_2 = \lambda^2 Ae^{\lambda t}$

$\Rightarrow (\lambda^2 + 341.421)Ae^{\lambda t} = 0$

$\Rightarrow \lambda = \pm i\,18.477$

& $\eta_{2,h} = Ae^{-18.47it} + Be^{18.47it}$ —— (vii)

let $\eta_{2,p} = De^{i(13)t} \Rightarrow \ddot{\eta}_{2,p} = -(13)^2 De^{i13t}$

& $\left[-13^2 + 341.421\right]D = +50$

$\Rightarrow D = .289 \simeq .29$

$\Rightarrow \eta_2 = Ae^{-18.47it} + Be^{18.47it} + .29e^{i(13)t}$

@ $t=0$ $\eta_2(0) = 0$

$\Rightarrow 0 = A + B + .29 \Rightarrow A + B = -.29$ — (viii)

$\dot{\eta}_2 = -18.47i\,Ae^{-i18.47i} + 18.47i\,Be^{18.47it} + 3.77i\,e^{i(13)t}$

@ $t=0$ $\dot{\eta}_2 = -2.82$

$-2.82 = -18.47i\,A + 18.47i\,B + 3.77i$

$-A + B = \dfrac{-2.82 - 3.77i}{18.47i} = -0.204 + .152i$ — (ix)

$$\Rightarrow \quad 2B = -.494 + .152i$$

$$B = -.247 + .076i$$

$$\& \quad A = -.043 - .076i$$

$$\therefore \eta_2 = (-.043 - .076i)e^{-18.471it} + (-.247 + .076i)e^{18.471it} + .29e^{i13t}$$

4. Physical Displacement

$$\begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{bmatrix} -.7071 & -.7071 \\ -0.5 & 0.5 \end{bmatrix} \begin{Bmatrix} \eta_1 \\ \eta_2 \end{Bmatrix}$$

$$x = -0.7071\eta_1 - .7071\eta_2$$

$$\& \quad \theta = -0.5\eta_1 + .5\eta_2.$$