Given:

Part A:

m=2kg , k=1200N/m , F_0=10N , ω =15rad/s , F(t)=F_0cos(ωt)

Part B:

F(t)=0 (ie free vibration)

modal damping ratio of 2%(ζ) for all modes

$x_1(0)=0.01m$; $x_2(0)=0m$; $x_3(0)=0m$; $\dot{x}_1(0)=0m/s$; $\dot{x}_2(0)=0m/s$; $\dot{x}_3(0)=0m/s$

Find:

Part A:

1. **Equations of motion;**
2. **Natural frequencies;**
3. **mode shapes;**
4. **orthonormal modal vectors;**
5. **orthonormal modal matrix;**
6. **obtain the steady state responses.**

Part B:

Find the steady state responses with the initial conditions.

Solve:

Part A:

(a) Equation of motion:

$$\begin{cases} 5m\ddot{x}_1 + 6kx_1 - kx_2 - 3kx_3 = F(t) \\ 2m\ddot{x}_2 + (x_2 - x_1)k = 0 \\ m\ddot{x}_3 + (x_3 - x_1)3k = 0 \end{cases}$$

Matrix form:

$$\begin{bmatrix} 5m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & m \end{bmatrix}\ddot{x} + \begin{bmatrix} 6k & -k & -3k \\ -k & k & 0 \\ -3k & 0 & 3k \end{bmatrix}x = \begin{bmatrix} F(t) \\ 0 \\ 0 \end{bmatrix}$$

(b) Free vibration $\Rightarrow$ frequency equation

$$\det\left[-\omega^2 m\begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} + k\begin{bmatrix} 6 & -1 & -3 \\ -1 & 1 & 0 \\ -3 & 0 & 3 \end{bmatrix}\right] = 0$$

set $\lambda = \dfrac{\omega^2 m}{k}$

$$\Rightarrow \det\begin{bmatrix} 6-5\lambda & -1 & -3 \\ -1 & 1-2\lambda & 0 \\ -3 & 0 & 3-\lambda \end{bmatrix} = 0$$

$\Rightarrow\ 10\lambda^3 - 47\lambda^2 + 38\lambda - 6 = 0$
$\Rightarrow \lambda_1$=3.7225 ; λ_2=0.21 ; λ_3=0.7675

$$\omega = \sqrt{\frac{k\lambda}{m}}$$

$\Rightarrow \omega_1$=47.2595 rad/s ; ω_2=11.2251 rad/s ; ω_3=21.4596 rad/s

(c) Substitute λ_i in the free vibration equations.
Mode shape vectors $\underline{x}_{(1)}$, $\underline{x}_{(2)}$ and $\underline{x}_{(3)}$ may be obtained as:

$$\begin{bmatrix} 6-5\lambda_i & -1 & -3 \\ -1 & 1-2\lambda_i & 0 \\ -3 & 0 & 3-\lambda_i \end{bmatrix} \begin{bmatrix} \underset{\sim}{x}_{1(i)} \\ \underset{\sim}{x}_{2(i)} \\ \underset{\sim}{x}_{3(i)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \underset{\sim}{x}_{(i)} = \begin{bmatrix} 1 \\ \dfrac{1}{1-2\lambda_i} \\ \dfrac{3}{3-\lambda_i} \end{bmatrix}$$

$$\Rightarrow \underset{\sim}{x}_{(1)} = \begin{bmatrix} 1 \\ -0.1552 \\ -4.1522 \end{bmatrix}; \underset{\sim}{x}_{(2)} = \begin{bmatrix} 1 \\ 1.7241 \\ 1.0753 \end{bmatrix}; \underset{\sim}{x}_{(3)} = \begin{bmatrix} 1 \\ -1.8692 \\ 1.3438 \end{bmatrix}$$

(d) Normalize the mass matrix:

$$\underset{\sim}{x}_{(i)}^T \cdot M \cdot \underset{\sim}{x}_{(i)} = M_{ii}$$

$$M_{11} = \begin{bmatrix} 1 & -0.1552 & -4.1522 \end{bmatrix} \begin{bmatrix} 5m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{bmatrix} 1 \\ -0.1552 \\ -4.1522 \end{bmatrix} = 44.5812$$

where m=2.
Similarly :

M_{22}=24.2034 ; M_{33}=27.5843

To get the orthonormal mode shape vectors , let $\tilde{\underset{\sim}{x}}_{(i)} = \dfrac{\underset{\sim}{x}_{(i)}}{\sqrt{M_{ii}}}$

$$\Rightarrow \tilde{\underset{\sim}{x}}_{(1)} = \begin{bmatrix} 0.1498 \\ -0.0232 \\ -0.6219 \end{bmatrix}; \tilde{\underset{\sim}{x}}_{(2)} = \begin{bmatrix} 0.2033 \\ 0.3505 \\ 0.2186 \end{bmatrix}; \tilde{\underset{\sim}{x}}_{(1)} = \begin{bmatrix} 0.1904 \\ -0.3559 \\ 0.2559 \end{bmatrix}$$

(e) The orthonormal modal matrix :

$$\tilde{P} = \begin{bmatrix} \tilde{x}_{(1)} & \tilde{x}_{(2)} & \tilde{x}_{(3)} \end{bmatrix} = \begin{bmatrix} 0.1498 & 0.2033 & 0.1904 \\ -0.0232 & 0.3505 & -0.3559 \\ -0.6219 & 0.2186 & -0.2559 \end{bmatrix}$$

It's easy to prove that $\tilde{P}^T \cdot M \cdot \tilde{P} = I$ and $\tilde{P}^T \cdot K \cdot \tilde{P} = \Lambda$

(f) To decouple equation of motion:

let $x = \tilde{P} \cdot y$ then:

$$M\tilde{P}\ddot{y} + K\tilde{P}y = F$$

Premultiply by $\tilde{P}^T$

$$\Rightarrow \quad \tilde{P}^T M\tilde{P}\ddot{y} + \tilde{P}^T K\tilde{P}y = \tilde{P}^T F$$

$$\text{or} \quad \ddot{y} + \Lambda y = \tilde{P}^T F = \underset{\sim}{F}$$

$$\begin{aligned} \underset{\sim}{F} &= \tilde{P}^T F \\ &= \begin{bmatrix} 0.1498 & -0.0232 & -0.6219 \\ 0.2033 & 0.3505 & 0.2186 \\ 0.1904 & -0.3559 & -0.2559 \end{bmatrix} \begin{bmatrix} 10 \\ 0 \\ 0 \end{bmatrix} \cos(15t) \\ &= \begin{bmatrix} 1.4977 \\ 2.0326 \\ 1.9040 \end{bmatrix} \cos(15t) \end{aligned}$$

$$\ddot{y}_i + \omega^2{}_i y_i = \underset{\sim}{F}_i \cos(15t)$$

$$\Rightarrow \begin{cases} \ddot{y}_1 + 47.2595^2 y_1 = 1.4977\cos(15t) \\ \ddot{y}_2 + 11.2251^2 y_2 = 2.0326\cos(15t) \\ \ddot{y}_3 + 21.4596^2 y_3 = 1.9040\cos(15t) \end{cases}$$

steady state solutions:

$$\Rightarrow \begin{cases} y_1 = \quad 0.0007\cos(15t) \\ y_2 = -0.0205\cos(15t) \\ y_3 = \quad 0.0081\cos(15t) \end{cases}$$

Transform back to get x via:

$$x = \tilde{P} \cdot y = \tilde{P} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\Rightarrow \begin{cases} x_1(t) = -0.0025\cos(15t) \\ x_2(t) = -0.0101\cos(15t) \\ x_3(t) = -0.0029\cos(15t) \end{cases}$$

<u>Part B:</u>

To decouple the equation of motion :

let $x = \tilde{P} \cdot y$ then:

$$M\tilde{P}\ddot{y} + C\tilde{P}\dot{y} + K\tilde{P}y = 0$$

Premultiply by $\tilde{P}^T$

$$\Rightarrow \quad \tilde{P}^T M\tilde{P}\ddot{y} + \tilde{P}^T C\tilde{P}\dot{y} + \tilde{P}^T K\tilde{P}y = \tilde{P}^T F$$

$$\text{or} \quad \ddot{y}_i + 2\zeta\omega_{ni}\dot{y}_i + \omega^2{}_{ni} y_i = 0$$

$$\Rightarrow \quad y_i = A_i e^{-\zeta\omega_{ni}t} \sin(\omega_{di} t + \phi_i); \omega_{di} = \omega_{ni}\sqrt{1-\zeta^2}$$

$$\text{and} \quad \dot{y}_i = A_i e^{-\zeta\omega_{ni}t}\omega_{di}\cos(\omega_{di}t + \phi_i) - A_i e^{-\zeta\omega_{ni}t}\zeta\omega_{ni}\sin(\omega_{di}t + \phi_i)$$

$$\Rightarrow \quad \begin{cases} y_1 = A_1 e^{-0.9452t}\sin(47.2503t + \phi_1) \\ y_2 = A_2 e^{-0.2245t}\sin(11.2228t + \phi_2) \\ y_3 = A_3 e^{-0.4292t}\sin(21.4553t + \phi_3) \end{cases}$$

$$\text{because} \begin{cases} x = \tilde{P} \cdot y \\ \dot{x} = \tilde{P} \cdot \dot{y} \end{cases} \Rightarrow \begin{cases} y = \tilde{P}^{-1} \cdot x \\ \dot{y} = \tilde{P}^{-1} \cdot \dot{x} \end{cases}$$

$$\text{So} \quad x(0) = \begin{bmatrix} 0.01 \\ 0 \\ 0 \end{bmatrix}; \dot{x}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad y(0) = \begin{bmatrix} 0.0150 \\ 0.0203 \\ 0.0190 \end{bmatrix}; \dot{y}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solve those equations and get the following:

$$\begin{cases} y_1 = 0.0150e^{-0.9452t}\sin(47.2503t + 1.5508) \\ y_2 = 0.0203e^{-0.2245t}\sin(11.2228t + 1.5508) \\ y_3 = 0.0190e^{-0.4292t}\sin(21.4553t + 1.5508) \end{cases}$$

$x = \tilde{P} \cdot y$ so get the steady responses:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} 0.1498 & 0.2033 & 0.1904 \\ -0.0232 & 0.3505 & -0.3559 \\ -0.6219 & 0.2186 & -0.2559 \end{bmatrix} \begin{bmatrix} 0.0150e^{-0.9452t}\sin(47.2503t + 1.5508) \\ 0.0203e^{-0.2245t}\sin(11.2228t + 1.5508) \\ 0.0190e^{-0.4292t}\sin(21.4553t + 1.5508) \end{bmatrix}$$

MATLAB code for Question 2:

```
%Part A
M=[5*m 0 0;0 2*m 0;0 0 m];
K=[6*k -1*k -3*k;-1*k 1*k 0;-3*k 0 3*k];
%M=[5 0 0;0 2 0;0 0 1];
%K=[6 -1 -3;-1 1 0;-3 0 3];

%Determine natural frequencies from equations
[v,d]=eig((M^-1)*K);
w=d^0.5

%Determine mode shapes
P(:,1)=v(:,1)/v(1,1);
P(:,2)=v(:,2)/v(1,2);
P(:,3)=v(:,3)/v(1,3)

%determine orthonormal modal vectors and modal matrix
M11=P(:,1)'*M*P(:,1);
M22=P(:,2)'*M*P(:,2);
M33=P(:,3)'*M*P(:,3);
Q(:,1)=P(:,1)/M11^0.5;
Q(:,2)=P(:,2)/M22^0.5;
Q(:,3)=P(:,3)/M33^0.5


%Decouple equations of motion and obtain steady state responses
F=[10 0 0]';
F=Q'*F
Y=F.*(inv(w^2-15*15*eye(3))*[1 1 1]')
X=Q*Y

%Part B
dampratio=0.02;
w_d=w*((1-dampratio^2)^0.5)*[1 1 1]';
dampratio*w*[1 1 1]';
x0=[0.01 0 0]';
xdot0=[0 0 0]';
y0=inv(Q)*x0;
ydot0=inv(Q)*xdot0;
theta=atan(w_d./(dampratio*w*[1 1 1]'))
A=y0./sin(theta)
```

Question 3(problem 6.7)

Given:

$w(x,0)=\sin\frac{3\pi x}{l}$; $w_t(x,0)=0$; $l=1.4m$; $m=110g$; $\tau=11.1\times10^4 N$

Find:

- **response of string;**
- **plot the response at** $x=\frac{l}{2}$ **and** $x=\frac{l}{4}$

Solve:

From the general solution:

$$w(x,t)=\sum_{n=1}^{\infty}(C_n\sin\omega_n t+D_n\cos\omega_n t)\sin\frac{n\pi x}{l}$$

Since the initial velocity , $w_t(x,0)=0$

$$C_k=0$$

$$D_k=\frac{2}{l}\int_0^l w(x,0)\sin\frac{k\pi x}{l}dx \qquad \text{for k=1,2,3……} \qquad (1)$$

$$\Rightarrow \quad w(x,t)=\sum_{n=1}^{\infty}D_n\sin\frac{n\pi x}{l}\cos\frac{n\pi c}{l}t \qquad \text{since: } \omega_n=\frac{n\pi c}{l} \qquad (2)$$

The initial condition $w(x,0)$ is given by:

$$w(x,0)=\sin\frac{3\pi x}{l} \qquad (3)$$

Substituting (3)in (1) so:

$$D_k=\frac{2}{l}\int_0^l\sin\frac{3\pi x}{l}\sin\frac{k\pi x}{l}dx \qquad \text{for k=1,2,3……}$$

if $k\neq3$ so, $D_k=0$, for for k=1,2,4,5……

only when k=3, $D_k\neq0$ for k=3

$$D_3=\frac{2}{l}\int_0^l\sin\frac{3\pi x}{l}\sin\frac{3\pi x}{l}dx \qquad (4)$$

$$\text{set } \overline{x}=\frac{\pi}{l}x \qquad (5)$$

substitute (5) in (4):

$$\begin{aligned}D_3&=\frac{2}{\pi}\int_0^{\pi}\sin3\overline{x}\sin3\overline{x}d\overline{x}\\&=\frac{2}{\pi}\int_0^{\pi}\sin^2 3\overline{x}d\overline{x}\\&=\frac{1}{\pi}\int_0^{\pi}1-\cos6\overline{x}d\overline{x}\\&=1\end{aligned}$$

so the response of the string is:

$$w(x,t)=\sum_{n=1}^{\infty} D_n \sin\frac{n\pi x}{l}\cos\frac{n\pi c}{l}t$$

$$= D_3 \sin\frac{3\pi x}{l}\cos\frac{3\pi c}{l}t$$

$$(\, l = 1.4m;\, \rho A = 100 gper 1.4m = 0.0786 kg/m \;\; ; \frac{\pi c}{l} = \frac{\pi}{l}\sqrt{\frac{\tau}{\rho A}} = 2666.69 rad/s\,)$$

$$w(x,t)=\sin 6.732x\cos 8000t$$

plot the response:

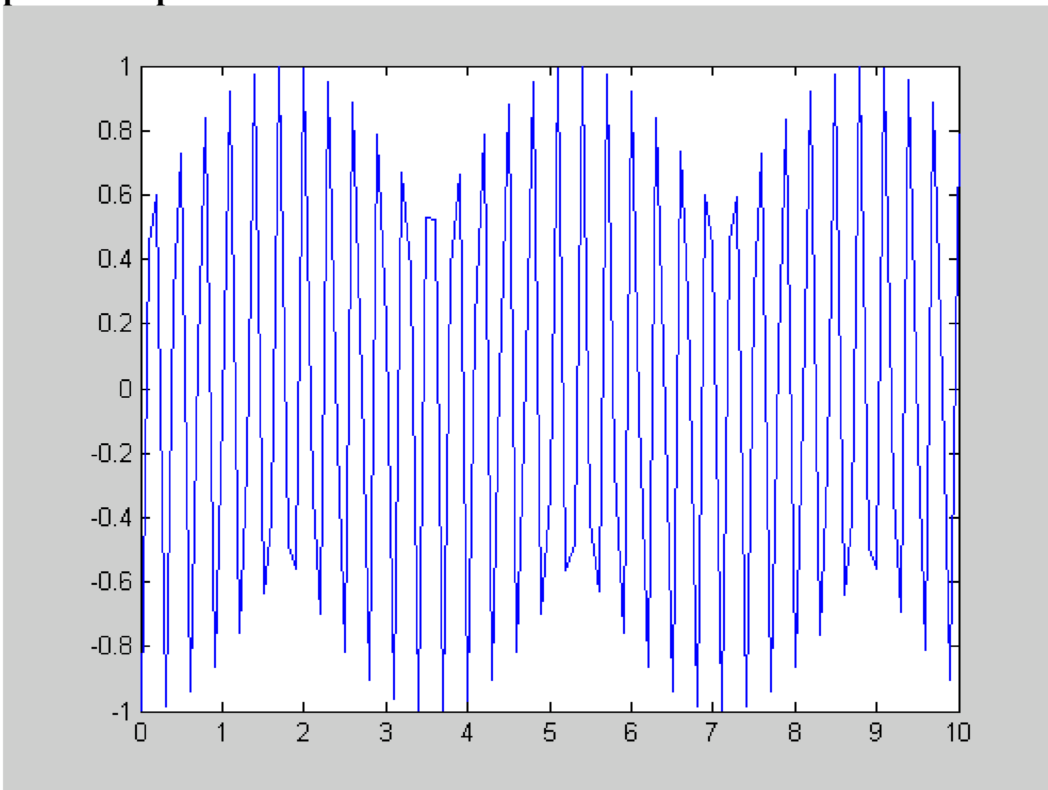

Figure 1: x=l/2

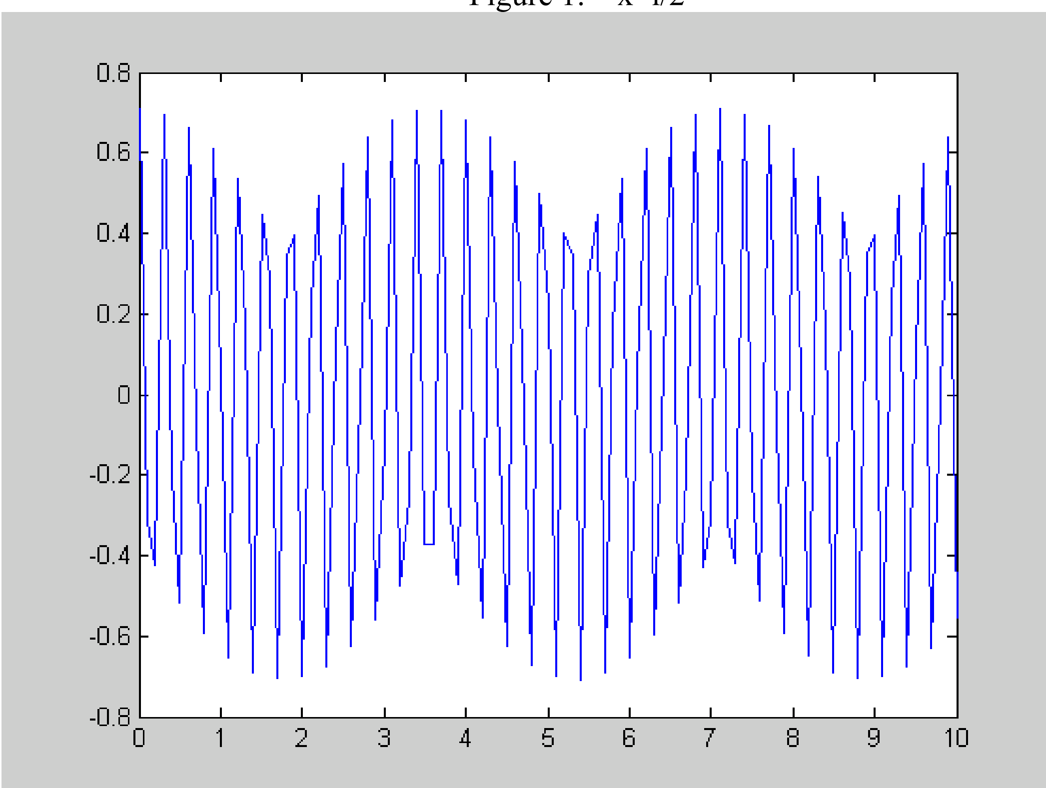

Figure 2: x=l/4