Q1:

a) Three degree of freedom.

b)

$$T = \frac{1}{2}m_1\dot{x}_1^{\,2} + \frac{1}{2}m_2\dot{x}_2^{\,2} + \frac{1}{2}m_3\dot{x}_3^{\,2}$$

$$U = \frac{1}{2}k_1x_1^{\,2} + \frac{1}{2}k_2(x_2-x_1)^2 + \frac{1}{2}k_3(x_2-x_3)^2 + \frac{1}{2}k_3(x_3-x_2)^2 + \frac{1}{2}k_3(x_2-x_3)^2$$

$$L = T - U$$

$$= \frac{1}{2}m_1\dot{x}_1^{\,2} + \frac{1}{2}m_2\dot{x}_2^{\,2} + \frac{1}{2}m_3\dot{x}_3^{\,2} - \frac{1}{2}k_1x_1^{\,2} - \frac{1}{2}k_2(x_2-x_1)^2 - \frac{1}{2}k_3(x_3-x_2)^2 - \frac{1}{2}k_3(x_2-x_3)^2 \;)$$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{x}_1}) = m_1\ddot{x}_1 \quad , \quad \frac{\partial L}{\partial x_1} = -k_1x_1 + k_2(x_2-x_1)$$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{x}_2}) = m_2\ddot{x}_2 \quad , \quad \frac{\partial L}{\partial x_2} = -k_2(x_2-x_1) - k_3(x_2-x_3) + k_3(x_3-x_2)$$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{x}_3}) = m_3\ddot{x}_3 \quad , \quad \frac{\partial L}{\partial x_3} = -k_3(x_3-x_2) + k_3(x_2-x_3)$$

$for \quad m_1 :$

$$m_1\ddot{x}_1 + (k_1+k_2)x_1 - k_2x_2 = 0$$

$for \quad m_2 :$

$$m_2\ddot{x}_2 + (k_2+2k_3)x_2 - k_2x_1 - 2k_3x_3 = 0$$

$for \quad m_3 :$

$$m_3\ddot{x}_3 + 2k_3x_3 - 2k_3x_2 = 0$$

$$\therefore \; the \quad equation \quad of \quad motion \quad is \Rightarrow \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} (k_1+k_2) & -k_2 & 0 \\ -k_2 & (k_2+2k_3) & -2k_3 \\ 0 & -2k_3 & 2k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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$c -$

$$\sum F_{x_1} = 0 \Rightarrow m_1\ddot{x}_1 + (k_1+k_2)x_1 - k_2x_2 = 0$$

$$\sum F_{x_2} = 0 \Rightarrow m_2\ddot{x}_2 + (k_2+2k_3)x_2 - k_2x_1 - 2k_3x_3 = 0$$

$$\sum F_{x_2} = 0 \Rightarrow m_3\ddot{x}_3 + 2k_3x_3 - 2k_3x_2 = 0$$

$$\therefore \; the \quad equation \quad of \quad motion \quad is \Rightarrow \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} (k_1+k_2) & -k_2 & 0 \\ -k_2 & (k_2+2k_3) & -2k_3 \\ 0 & -2k_3 & 2k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

d) the system is statically coupled.

**Q2:**

$$T = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2 + \frac{1}{2} j \dot{\theta}_1^2 + \frac{1}{2} j \dot{\theta}_2^2$$

$$T = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2 + \frac{1}{2}\left(\frac{m r^2}{2}\right)\left(\frac{\dot{x}_1}{r}\right)^2 + \frac{1}{2}\left(\frac{m r^2}{2}\right)\left(\frac{\dot{x}_2}{r}\right)^2$$

$$T = \frac{3}{4} m \dot{x}_1^2 + \frac{3}{4} m \dot{x}_2^2$$

$$U = \frac{1}{2} k \left(x_1 - x_2\right)^2$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} + \frac{\partial U}{\partial x} + \frac{\partial D}{\partial \dot{x}}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right) = \frac{3}{2} m \ddot{x}_1$$

$$\frac{\partial U}{\partial x_1} = k x_1 - k x_2$$

$$\Rightarrow \frac{3}{2} m \ddot{x}_1 + k x_1 - k x_2 = 0$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_2}\right) = \frac{3}{2} m \ddot{x}_2$$

$$\frac{\partial U}{\partial x_2} = -k x_1 + k x_2$$

$$\Rightarrow \frac{3}{2} m \ddot{x}_2 - k x_1 + k x_2 = 0$$

Matrix Form:

$$\begin{bmatrix} \frac{3}{2} m & 0 \\ 0 & \frac{3}{2} m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\det = \begin{bmatrix} k - \omega^2 \frac{3}{2} m & -k \\ -k & k - \omega^2 \frac{3}{2} m \end{bmatrix} = 0$$

$$k^2 - 3\omega^2 mk + \frac{9}{4}\omega^4 m^2 - k^2 = 0$$

$$\omega^2 \left[ (\frac{9}{4} m^2)\omega^2 - (3mk) \right] = 0$$

$$\omega_1^2 = 0, \text{ and } \omega_2^2 = \frac{4k}{3m}$$

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Eigenvecors:

For $\omega_1$=0:

$$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Normalaize the eigenvector with respect to mass matrix:

$$\alpha^2 \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} (3/2)m & 0 \\ 0 & (3/2)m \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = 1$$

$$\alpha^2 \; v_1^T [M] \, v_1 = 1$$

$$\alpha^2 (3m) = 1, \quad \text{or} \quad \alpha = \frac{1}{\sqrt{3m}}$$

$$V_1 = \alpha v_1 = \frac{1}{\sqrt{3m}} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} \frac{1}{\sqrt{3m}} \\ \frac{1}{\sqrt{3m}} \end{Bmatrix}$$

For $\omega_2 = \sqrt{\frac{4k}{3m}}$:

$$\begin{bmatrix} -k & -k \\ -k & -k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$kx_1 + kx_2 = 0$$

$let,$

$$x_1 = 1$$

$$x_2 = \frac{-k}{k} = -1$$

$$v_2 = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

$$\alpha^2 \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} (3/2)m & 0 \\ 0 & (3/2)m \end{bmatrix} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} = 1$$

$$\alpha^2 \begin{bmatrix} (3/2)m & -(3/2)m \end{bmatrix} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} = 1$$

$$\alpha^2 (3m) = 1$$

$$\alpha = \frac{1}{\sqrt{3m}}$$

$$v_2 = \begin{Bmatrix} \frac{1}{\sqrt{3m}} \\ \frac{-1}{\sqrt{3m}} \end{Bmatrix}$$

Modal matrix:

$$\left[\tilde{U}\right] = \frac{1}{\sqrt{3m}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Now, let *M*=10 *kg* and *k*=300 *N/m*,

$$\left[\tilde{U}\right] = \frac{1}{\sqrt{30}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \text{ and } \left[M\right] = \begin{bmatrix} 15 & 0 \\ 0 & 15 \end{bmatrix}$$

$$\begin{Bmatrix} q_1(0) \\ q_2(0) \end{Bmatrix} = \left[\tilde{U}^T\right] \left[M\right] \begin{Bmatrix} x_1(0) \\ x_2(0) \end{Bmatrix}$$

$$\begin{Bmatrix} q_1(0) \\ q_2(0) \end{Bmatrix} = \frac{1}{\sqrt{30}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 15 & 0 \\ 0 & -15 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} q_1(0) \\ q_2(0) \end{Bmatrix} = \frac{1}{\sqrt{30}}\begin{Bmatrix} 15 \\ 15 \end{Bmatrix}$$

$$\begin{Bmatrix} \dot{q}_1(0) \\ \dot{q}_2(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$q_1(t) = q_1(0)\cos(\omega_1 t) + \frac{\dot{q}_1(0)}{\omega_1}\sin(\omega_1 t)$$

$$q_1(t) = \frac{15}{\sqrt{30}}$$

$$q_2(t) = q_2(0)\cos(\omega_2 t) + \frac{\dot{q}_2(0)}{\omega_2}\sin(\omega_2 t)$$

$$q_2(t) = \frac{15}{\sqrt{30}}\cos(\sqrt{\frac{4k}{2m}}t)$$

$$q_2(t) = \frac{15}{\sqrt{30}}\cos(\sqrt{60}t)$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \tilde{U} \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix}$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \frac{1}{\sqrt{30}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix}$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \frac{1}{\sqrt{30}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{15}{\sqrt{30}} \\ \frac{15}{\sqrt{30}}\cos(\sqrt{60}t) \end{bmatrix}$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \frac{1}{\sqrt{30}}\begin{bmatrix} \frac{15}{\sqrt{30}} + \frac{15}{\sqrt{30}}\cos(\sqrt{60}t) \\ \frac{15}{\sqrt{30}} - \frac{15}{\sqrt{30}}\cos(\sqrt{60}t) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + \frac{1}{2}\cos(\sqrt{60}t) \\ \frac{1}{2} - \frac{1}{2}\cos(\sqrt{60}t) \end{bmatrix}$$