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Mechanical Vibrations
MENG 470
Final Exam

Closed Book Exam
Time: 2 Hours
Monday: 19/4/1425 H

PART I

Select the most appropriate answer from the multiple choices given:

1. Which one of the following is a valid application for the Principle of Virtual Work?
   a. Solve an eigenvalue problem for a system
   b. Find the static equilibrium of a system
   c. Find the dynamic equilibrium of a system
   d. Determine the stability of a system

2. D'Alembert's Principle can be used directly to derive the differential equations of motion for a dynamic system.
   a. True
   b. False

3. A single-degree-of-freedom system has only one natural frequency.
   a. True
   b. False

4. The real part of the solution to the characteristic equation for a single-degree-of-freedom system is zero. Which one of the following best describes the system?
   a. Underdamped
   b. Unstable
   c. Undamped
   d. Nonperiodic

7. All of the normal modes of a multi-degree-of-freedom system are orthogonal, except for rigid body modes.
   a. True
   b. False

8. If a multi-degree-of-freedom system is positive semi-definite, which one of the following is not true?
   a. The stiffness matrix is positive semi-definite
   b. The eigenvalues of the system cannot be determined
   c. The system has at least one rigid body mode
   d. The system has at least one eigenvalue with a value of zero

9. If the modes, *u*, of a multi-degree-of-freedom system are normalized with respect to the mass matrix, [m], the expression $[u]^T[m][u]$ yields the identity matrix.
   a. True
   b. False

10. Which one of the following is not a characteristic of a continuous dynamic system?
    a. The equations of motion are partial differential equations
    b. The system response cannot be calculated
    c. The system has an infinite number of degrees of freedom
    d. Both boundary conditions and initial conditions must be specified

11. Which one of the following is not required to perform a modal analysis of a continuous system?
    a. Calculate the natural frequencies
    b. Calculate the natural modes
    c. Normalize the natural modes
    d. Calculate Rayleigh's Quotient

12. The flexibility and the stiffness matrices are the inverse of one another.
    a. True
    b. False

13. The element stiffness matrices are always singular unless the boundary conditions are applied.
    a. True
    b. False

14. The finite element method is:
    a. an approximate analytical method
    b. a numerical method
    c. an exact analytical method

15. What cause whirling of rotating shafts?
    a. Mass unbalance
    b. Fluid friction in the bearings
    c. Gyroscopic forces
    d. All above.

16. To measure mechanical vibrations, we use:
    a. Accelerometers and signal analyzer.
    b. Sound level meters.
    c. Exciter and exciter controller.
    d. All above.

17. The fundamental natural frequency of a system is a:

a. The largest value
b. The smallest value
c. Any value.

18. When a two degree of freedom system is subjected to a harmonic force, the system vibrates at the:

a. Frequency of applied force.
b. Smaller natural frequency
c. Larger natural frequency.

19. The response of an undamped system under resonance will be:

a. Very large.
b. Infinity.
c. Zero.

20. Gibbs phenomenon denotes an anomalous behavior is the Fourier series representation of a:

a. Harmonic function.
b. Periodic function.
c. Random function

PART II

Q1.

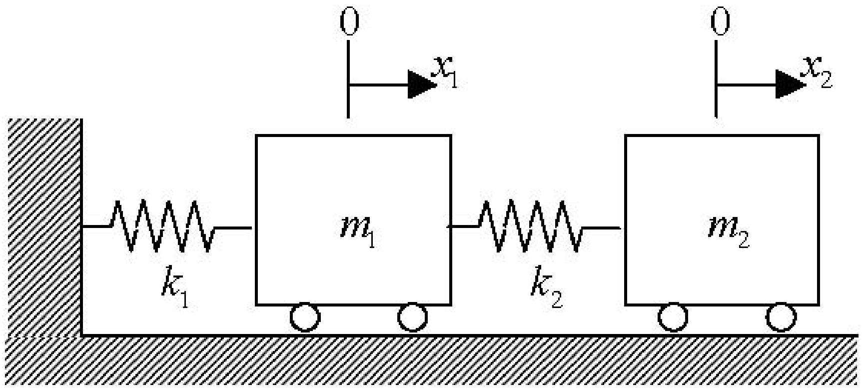


Figure 2

Solutions

(a)

Equations of motion

$m_1\ddot{x}_1 = -k_1x_1 - k_2(x_1 - x_2)$

$m_2\ddot{x}_2 = -k_2(x_2 - x_1)$

Equations of motion in matrix form

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The substitution of numerical values yields

$$\begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or

$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0}$

where

$\mathbf{M} = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}$ $\mathbf{K} = \mathbf{K} = \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix}$ and $\mathbf{x}$ and . $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Assume the solution as . Then, and . Substitution of these yields

$\mathbf{x} = \mathbf{u}e^{j\omega_n t}$. $\dot{\mathbf{x}} = j\omega_n\mathbf{u}e^{j\omega_n t}$ a$\ddot{\mathbf{x}} = -\omega_n^{\ 2}\mathbf{u}e^{j\omega_n t}$. S

$$\begin{bmatrix} -9\lambda + 27 & -3 \\ -3 & -\lambda + 3 \end{bmatrix}\mathbf{u} = \mathbf{0}$$

where . The characteristic equation of the system is $\lambda = \omega_n{}^2$.

$$\det\begin{bmatrix} -9\lambda+27 & -3 \\ -3 & -\lambda+3 \end{bmatrix} = 0$$

$$(-9\lambda+27)(-\lambda+3)-3\cdot 3=0$$

$$9\lambda^2-(1\cdot 27+9\cdot 3)\lambda+27\cdot 3-9=0$$

$$9\lambda^2-54\lambda+72=0$$

$$\lambda^2-6\lambda+8=0$$

$$(\lambda-2)(\lambda-4)=0$$

λ = 2, 4

So, the natural frequencies are

$\omega_{n1}=\sqrt{2}$, $\omega_{n2}=\sqrt{4}=2$

(b)

Let modes be **u** for $_1\omega_{n1}=\sqrt{2}$ and for . $\mathbf{u}_2\,\omega_{n2}=2$

λ $\omega_n=\omega_{n1}=\sqrt{2}$)

The equations becomes

$$\begin{bmatrix} -9\cdot 2+27 & -3 \\ -3 & -2+3 \end{bmatrix}\mathbf{u}_1=\mathbf{0}$$

$$\begin{bmatrix} 9 & -3 \\ -3 & 1 \end{bmatrix}\mathbf{u}_1=\mathbf{0}$$

$$\begin{cases} 9u_{11}-3u_{12}=0 \\ -3u_{11}+u_{12}=0 \end{cases}$$

λ $\omega_n=\omega_{n2}=2$)

The equations becomes

$$\begin{bmatrix} -9\cdot 4+27 & -3 \\ -3 & -4+3 \end{bmatrix}\mathbf{u}_1=\mathbf{0}$$

$$\begin{bmatrix} -9 & -3 \\ -3 & -1 \end{bmatrix}\mathbf{u}_1=\mathbf{0}$$

$$\begin{cases} -9u_{21}-3u_{22}=0 \\ -3u_{21}-u_{22}=0 \end{cases}$$

(c)

the modal equations yield $_{11}=\frac{1}{3}u_{12}$ *u* and $u_{21}=-\frac{1}{3}u_{22}$. Let = = 1. Then, $u_{12}u_{22}$

$$\mathbf{u}_1 = \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}.$$

, .

The general solution becomes

$$\mathbf{x}(t) = A_1 \mathbf{u}_1 \sin\left(\omega_{n1} t + \phi_1\right) + A_2 \mathbf{u}_2 \sin\left(\omega_{n2} t + \phi_2\right)$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A_1 \begin{bmatrix} 1/3 \\ 1 \end{bmatrix} \sin\left(\sqrt{2}t + \phi_1\right) + A_2 \begin{bmatrix} -1/3 \\ 1 \end{bmatrix} \sin\left(2t + \phi_2\right)$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} A_1 \sin\left(\sqrt{2}t + \phi_1\right) - \frac{1}{3} A_2 \sin\left(2t + \phi_2\right) \\ A_1 \sin\left(\sqrt{2}t + \phi_1\right) + A_2 \sin\left(2t + \phi_2\right) \end{bmatrix}$$

and its velocity:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \omega_{n1} A_1 \cos\left(\sqrt{2}t + \phi_1\right) - \frac{1}{3} \omega_{n2} A_2 \cos\left(2t + \phi_2\right) \\ \omega_{n1} A_1 \cos\left(\sqrt{2}t + \phi_1\right) + \omega_{n2} A_2 \cos\left(2t + \phi_2\right) \end{bmatrix}$$

By substituting the initial conditions, four equations rise

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} A_1 \sin\phi_1 - \frac{1}{3} A_2 \sin\phi_2 \\ A_1 \sin\phi_1 + A_2 \sin\phi_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{->} \begin{cases} A_1 \sin\phi_1 - A_2 \sin \\ A_1 \sin\phi_1 + A_2 \sin \end{cases} \text{->} \begin{cases} A_1 \sin\phi_1 - A_2 \sin\phi_2 = 3 \\ A_1 \sin\phi_1 + A_2 \sin\phi_2 = 0 \end{cases}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \sqrt{2} A_1 \cos\phi_1 - \frac{2}{3} A_2 \cos\phi_2 \\ \sqrt{2} A_1 \cos\phi_1 + 2 A_2 \cos\phi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{->} \begin{cases} \sqrt{2} A_1 \cos\phi_1 - 2 A_2 \text{cc} \\ \sqrt{2} A_1 \cos\phi_1 + 2 A_2 \text{cc} \end{cases} \text{->}$$

$$\begin{cases} \sqrt{2} A_1 \cos\phi_1 - 2 A_2 \cos\phi_2 = 0 \\ \sqrt{2} A_1 \cos\phi_1 + 2 A_2 \cos\phi_2 = 0 \end{cases}$$

where there are four parameters: , , , . Adding the third and fourth equations yields $A_1 A_2 \phi_1 \phi_2$

$$2\sqrt{2} A_1 \cos\phi_1 = 0$$

so that . Therefore, the fours equation reduces to $\phi_1 = \pi / 2$.

$$2 A_2 \cos\phi_2 = 0$$

so that . Substitution of the values of and into the first and second equations yields

$\phi_2 = \pi / 2$. $\phi_1 \phi_2$

$$\begin{cases} A_1 - A_2 = 3 \\ A_1 + A_2 = 0 \end{cases}$$

which has solutions =3/2 and =-3/2. Thus $A_1 A_2$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \cdot \frac{3}{2} \sin\left(\sqrt{2}t + \frac{\pi}{2}\right) - \frac{1}{3} \cdot \left(-\frac{3}{2}\right) \sin\left(2t + \frac{\pi}{2}\right) \\ \frac{3}{2} \sin\left(\sqrt{2}t + \frac{\pi}{2}\right) + \frac{3}{2} \sin\left(2t + \frac{\pi}{2}\right) \end{bmatrix} = \begin{bmatrix} 0.5 \cos\sqrt{2}t + 0.5 \cos 2t \\ 1.5 \cos 2t - 1.5 \cos 2t \end{bmatrix}$$

(d)

Uncoupled equations of motion is

$$\begin{cases} \ddot{\eta}_1 + \omega_{n1}{}^2 \eta_1 = 0 \\ \ddot{\eta}_2 + \omega_{n2}{}^2 \eta_2 = 0 \end{cases} \quad -> \quad \begin{cases} \ddot{\eta}_1 + 2\eta_1 = 0 \\ \ddot{\eta}_2 + 4\eta_2 = 0 \end{cases}$$

We want to simulate this uncoupled system with initial conditions , for the coupled system. Uncoupled initial conditions are , , so we will find the modal matrix first. Modal vectors can be

$\mathbf{x}(0) = \mathbf{x}_0$, $\dot{\mathbf{x}}(0) = \dot{\mathbf{x}}_0$

represented as $_0 = \mathbf{U}^T \mathbf{M} \dot{\mathbf{x}}_0$, so we $\boldsymbol{\eta}_0 = \mathbf{U}^T \mathbf{M} \mathbf{x}_0 \dot{\boldsymbol{\eta}} \mathbf{U}$

$$\mathbf{u}_1 = \begin{bmatrix} u_1 \\ 3u_1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} u_2 \\ -3u_2 \end{bmatrix}$$

Thus, =3, =-3. Conditions for orthonormality is **u**, so that orthonormal modal vectors are thus

$$a_1 a_2 {}^T_i \mathbf{M} \mathbf{u}_i = \begin{bmatrix} u_i & a_i u_i \end{bmatrix} \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} u_i \\ a_i u_i \end{bmatrix} = 1$$

$$\mathbf{u}_1 = \begin{bmatrix} \dfrac{1}{\sqrt{m_1 + m_2 a_1{}^2}} \\ \dfrac{a_1}{\sqrt{m_1 + m_2 a_1{}^2}} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{\sqrt{9 + 1 \cdot 3^2}} \\ \dfrac{3}{\sqrt{9 + 1 \cdot 3^2}} \end{bmatrix} = \begin{bmatrix} 0.2357 \\ 0.7071 \end{bmatrix}, \quad \mathbf{u}_2 = \quad {}_2 = \begin{bmatrix} \dfrac{1}{\sqrt{m_1 + m_2 a_2{}^2}} \\ \dfrac{a_2}{\sqrt{m_1 + m_2 a_2{}^2}} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{\sqrt{9 + 1 \cdot (-3)^2}} \\ \dfrac{-3}{\sqrt{9 + 1 \cdot (-3)^2}} \end{bmatrix} = \begin{bmatrix} 0.2357 \\ -0.7071 \end{bmatrix}$$

, **u**

Modal matrix is therefore

$$\mathbf{U} = \begin{bmatrix} 0.2357 & 0.2357 \\ 0.7071 & -0.7071 \end{bmatrix}$$ (Check **U**) $${}^T \mathbf{M} \mathbf{U} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Uncoupled initial conditions are

$$\boldsymbol{\eta}_0 = \mathbf{U}^T \mathbf{M} \mathbf{x}_0 = \begin{bmatrix} 0.2357 & 0.7071 \\ 0.2357 & -0.7071 \end{bmatrix} \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.1213 \\ 1.1213 \end{bmatrix}$$

$$\dot{\boldsymbol{\eta}}_0 = \mathbf{U}^T \mathbf{M} \dot{\mathbf{x}}_0 = \begin{bmatrix} 0.2357 & 0.7071 \\ 0.2357 & -0.7071 \end{bmatrix} \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Then

$$\eta_1(t) = \frac{\sqrt{\omega_{n1}{}^2 \eta_{01}{}^2 + \dot{\eta}_{01}{}^2}}{\omega_{n1}} \sin\left(\omega_{n1} t + \tan^{-1} \frac{\omega_{n1} \eta_{01}}{\dot{\eta}_{01}}\right) = \frac{\sqrt{2 \cdot 2.1213^2 + 0^2}}{\sqrt{2}} \sin\left(\sqrt{2} t + \tan^{-1} \frac{\sqrt{2} \cdot 2.1213}{0}\right)$$

$$= 2.1213 \sin\left(\sqrt{2} t + \frac{\pi}{2}\right)$$

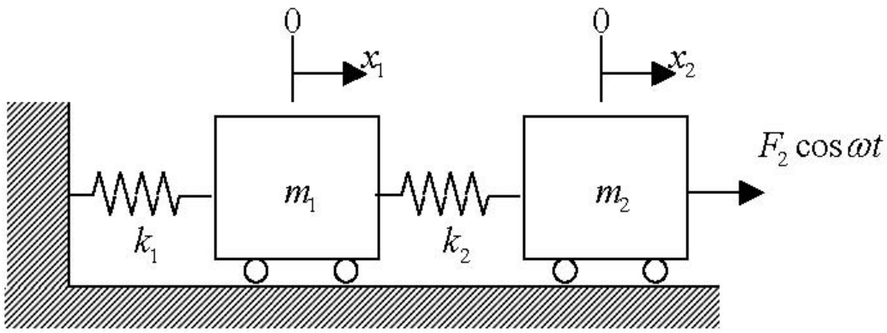
$$\eta_2(t) = \frac{\sqrt{\omega_{n2}{}^2 \eta_{02}{}^2 + \dot{\eta}_{02}{}^2}}{\omega_{n2}} \sin\left(\omega_{n2} t + \tan^{-1} \frac{\omega_{n2} \eta_{02}}{\dot{\eta}_{21}}\right) = \frac{\sqrt{2^2 \cdot 2.1213^2 + 0^2}}{2} \sin\left(2t + \tan^{-1} \frac{2 \cdot 2.1213}{0}\right)$$

$$= 2.1213 \sin\left(2t + \frac{\pi}{2}\right)$$

Finally,

$$\mathbf{x}(t) = \mathbf{U}\boldsymbol{\eta}(t) = \begin{bmatrix} 0.2357 & 0.2357 \\ 0.7071 & -0.7071 \end{bmatrix} \begin{bmatrix} 2.1213\sin\left(\sqrt{2}t + \dfrac{\pi}{2}\right) \\ 2.1213\sin\left(2t + \dfrac{\pi}{2}\right) \end{bmatrix} = \begin{bmatrix} 0.5\cos\sqrt{2}t + 0.5\cos 2t \\ 1.5\cos\sqrt{2}t - 1.5\cos 2t \end{bmatrix}$$

e)



The force vector is given by

$$\mathbf{F} = \begin{bmatrix} 0 \\ F_2 \end{bmatrix} \cos\omega t$$

(i)

Because the systems are undamped, solutions can be assumed as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \cos\omega t$$

Substituting this into the differential equation, we obtain

$$\begin{bmatrix} -\omega^2 m_1 + k_1 + k_2 & -k_2 \\ -k_2 & -\omega^2 m_2 + k_2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ F_2 \end{bmatrix}$$

$$\begin{bmatrix} 9(-\omega^2 + 3) & -3 \\ -3 & -\omega^2 + 3 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ F_2 \end{bmatrix}$$

or $$\mathbf{H} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ F_2 \end{bmatrix}$$

We hence obtain

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \mathbf{H}^{-1} \begin{bmatrix} 0 \\ F_2 \end{bmatrix} = \frac{1}{9(\omega^2 - 2)(\omega^2 - 4)} \begin{bmatrix} 9(-\omega^2 + 3) & -3 \\ -3 & -\omega^2 + 3 \end{bmatrix} \begin{bmatrix} 0 \\ F_2 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{3(\omega^2 - 2)(\omega^2 - 4)} F_2 \\ \dfrac{-\omega^2 + 3}{(\omega^2 - 2)(\omega^2 - 4)} F_2 \end{bmatrix}$$

(ii)

Forces in the uncoupled systems become

$$\mathbf{N} = \mathbf{U}^T\mathbf{F} = \begin{bmatrix} 0.2357 & 0.7071 \\ 0.2357 & -0.7071 \end{bmatrix} \begin{bmatrix} 0 \\ F_2 \cos 2t \end{bmatrix} = \begin{bmatrix} 0.7071 F_2 \cos 2t \\ -0.7071 F_2 \cos 2t \end{bmatrix}$$

The uncoupled equations of motion are therefore

$$\begin{cases} \ddot{\eta}_1 + 2\eta_1 = 0.7171 F_2 \cos 2t \\ \ddot{\eta}_2 + 4\eta_2 = -0.7171 F_2 \cos 2t \end{cases}$$

The steady-state solutions are

$$\eta_1(t) = \frac{f_1}{{\omega_n}^2 - \omega^2} \cos \omega t = \frac{0.7071 F_2}{2 - \omega^2} \cos \omega t$$

$$\eta_2(t) = \frac{f_2}{{\omega_n}^2 - \omega^2} \cos \omega t = \frac{0.7071 F_2}{4 - \omega^2} \cos \omega t$$

$$\mathbf{x}(t) = \mathbf{U}\boldsymbol{\eta}(t) = \begin{bmatrix} 0.2357 & 0.2357 \\ 0.7071 & -0.7071 \end{bmatrix} \begin{bmatrix} \dfrac{0.7071 F_2}{2 - \omega^2} \cos \omega t \\ \dfrac{0.7071 F_2}{4 - \omega^2} \cos \omega t \end{bmatrix} = \begin{bmatrix} \dfrac{1}{3(2-\omega^2)(4-\omega^2)} \cos \omega t \\ \dfrac{-\omega^2 + 3}{(2-\omega^2)(4-\omega^2)} \cos \omega t \end{bmatrix}$$

Therefore

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{3(2-\omega^2)(4-\omega^2)} F_2 \\ \dfrac{-\omega^2 + 3}{(2-\omega^2)(4-\omega^2)} F_2 \end{bmatrix}$$

Q2.

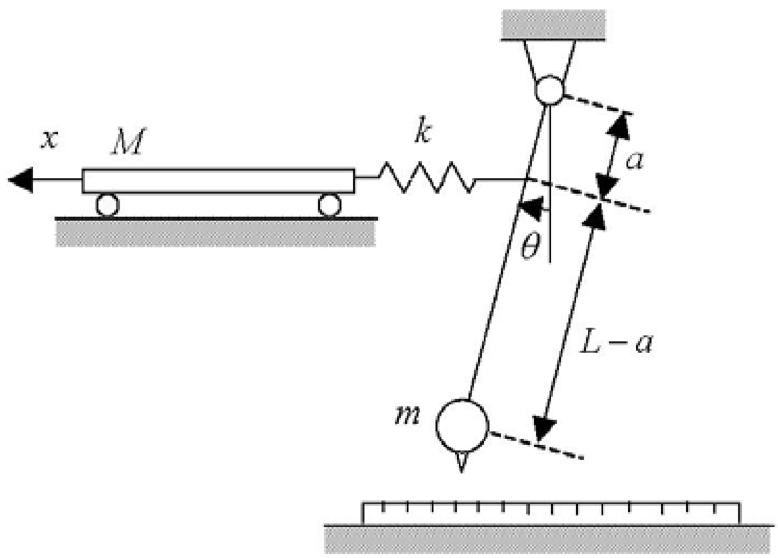


Figure 3

Answer:
Parameters

M=1 kg $\quad$ L=1 m
m=4 kg $\quad$ a=0.2 m
k=10 N/m

(a)
The equation of motion is given by

$$M\ddot{x} + k(x - a\theta) = 0$$

$$mL^2\ddot{\theta} + mgL\theta - ak(x - a\theta) = 0 \;\; -> \;\; mL^2\ddot{\theta} + \;\; -> \;\; mL^2\ddot{\theta} + (mgL + a^2k)\theta - akx = 0$$

Consequently,

$$\begin{bmatrix} M & 0 \\ 0 & mL^2 \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} k & -ka \\ -ka & mgL + a^2k \end{bmatrix}\begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} 10 & -2 \\ -2 & 39.6 \end{bmatrix}\begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(b)
Assume and θ. Then $x = X\cos\omega t = \Theta\cos\omega t$

$$\begin{bmatrix} -\omega^2 + 10 & -2 \\ -2 & -4\omega^2 + 39.6 \end{bmatrix}\begin{bmatrix} X \\ \Theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Then

$$\left(-\omega^2+10\right)\left(-4\omega^2+39.6\right)-4=0$$

$$\left(-\omega^2+10\right)\left(-\omega^2+9.9\right)-1=0$$

$$\omega^4-19.9\omega^2+98=0$$

ω^2 =8.95, 10.95 $\lambda =$

Natural frequencies

ω =2.99, 3.31

i) =8.95 λ_1

$$\begin{bmatrix} 1.05 & -2 \\ -2 & 3.80 \end{bmatrix}\begin{bmatrix} X \\ \Theta \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$X-1.90\Theta=0$$

$$\Theta=0.53X$$

Orthnormality condition

$$\mathbf{u}_1{}^T\mathbf{M}\mathbf{u}_1=\begin{bmatrix} u_1 & 0.53u_1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} u_1 \\ 0.53u_1 \end{bmatrix}=1$$

$$\mathbf{u}_1=\begin{bmatrix} 0.69 \\ 0.36 \end{bmatrix}$$

ii) =8.95 λ_1

$$\begin{bmatrix} -0.95 & -2 \\ -2 & -4.2 \end{bmatrix}\begin{bmatrix} X \\ \Theta \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$X+2.10\Theta=0$$

$$\Theta=-0.48X$$

Similarly,

$$\mathbf{u}_2=\begin{bmatrix} 0.72 \\ -0.35 \end{bmatrix}$$

$$\mathbf{U}=\begin{bmatrix} 0.69 & 0.72 \\ 0.36 & -0.35 \end{bmatrix}$$

(c)

$$x(0)=0, \dot{x}(0)=\dot{x}_0, \theta(0)=0, \dot{\theta}(0)=\dot{x}_0/L$$

$$\boldsymbol{\eta}_0=\mathbf{U}^T\mathbf{M}\mathbf{x}_0=\begin{bmatrix} 0.69 & 0.36 \\ 0.72 & -0.35 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\dot{\boldsymbol{\eta}}_0=\mathbf{U}^T\mathbf{M}\dot{\mathbf{x}}_0=\begin{bmatrix} 0.69 & 0.36 \\ 0.72 & -0.35 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} \dot{x}_0 \\ \dot{x}_0 \end{bmatrix}=\begin{bmatrix} 0.69 & 1.44 \\ 0.72 & -1.40 \end{bmatrix}\begin{bmatrix} \dot{x}_0 \\ \dot{x}_0 \end{bmatrix}=\begin{bmatrix} 2.13\dot{x}_0 \\ -0.68\dot{x}_0 \end{bmatrix}$$

Then

$$\eta_1(t) = \frac{\sqrt{{\omega_{n1}}^2{\eta_{01}}^2 + {\dot{\eta}_{01}}^2}}{\omega_{n1}}\sin\left(\omega_{n1}t + \tan^{-1}\frac{\omega_{n1}\eta_{01}}{\dot{\eta}_{01}}\right) = \frac{\sqrt{8.95\cdot 0 + (2.13\dot{x}_0)^2}}{2.99}\sin\left(2.99t + \tan^{-1}\frac{2.99\cdot 0}{2.13\dot{x}_0}\right)$$
$$= 0.71\dot{x}_0\sin(2.99t)$$

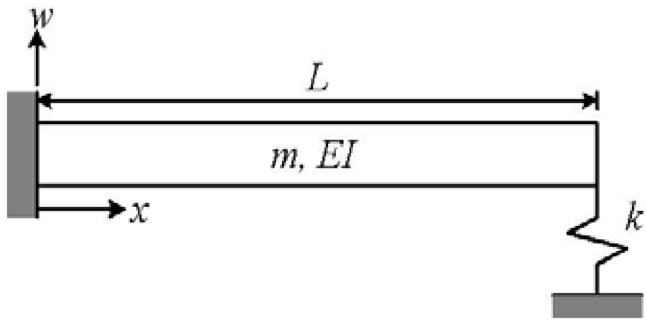
$$\eta_2(t) = \frac{\sqrt{{\omega_{n2}}^2{\eta_{02}}^2 + {\dot{\eta}_{02}}^2}}{\ } \sin\left(\omega_{n2} t + \tan^{-1}\frac{\omega_{n2}\eta_{02}}{\ }\right) = \frac{\sqrt{10.95\cdot 0 + (-0.68\dot{x}_0)^2}}{\ }\sin\left(3.31t + \tan^{-1}\frac{3.31\cdot 0}{\ }\right.$$

$$\eta_2(t) = \frac{\sqrt{{\omega_{n2}}^2{\eta_{02}}^2 + {\dot{\eta}_{02}}^2}}{\omega_{n2}}\sin\left(\omega_{n2}t + \tan^{-1}\frac{\omega_{n2}\eta_{02}}{\dot{\eta}_{21}}\right) = \frac{\sqrt{10.95\cdot 0 + (-0.68\dot{x}_0)^2}}{3.31}\sin\left(3.31t + \tan^{-1}\frac{3.31\cdot 0}{-0.68\dot{x}_0}\right)$$
$$= 0.21\dot{x}_0\sin(3.31t)$$

Finally,

$$\mathbf{x}(t) = \mathbf{U}\boldsymbol{\eta}(t) = \begin{bmatrix} 0.69 & 0.72 \\ 0.36 & -0.35 \end{bmatrix}\begin{bmatrix} 0.71\dot{x}_0\sin(2.99t) \\ 0.21\dot{x}_0\sin(3.31t) \end{bmatrix} = \begin{bmatrix} \dot{x}_0\{0.49\sin(2.99t) + 0.15\sin(3.31t)\} \\ \dot{x}_0\{0.26\sin(2.99t) - 0.07\sin(3.31t)\} \end{bmatrix}$$

Q3.



Boundary conditions

$$w(0) = 0$$

$$w'(0) = 0$$

$$w''(L) = 0$$

$$EIw'''(L) = kw(L)$$

Characteristic equation

$$w(x) = A\sin\sqrt{\beta}x + B\cos\sqrt{\beta}x + C\sinh\sqrt{\beta}x + D\cosh\sqrt{\beta}x$$

$$w'(x) = \sqrt{\beta}\left(A\cos\sqrt{\beta}x - B\sin\sqrt{\beta}x + C\cosh\sqrt{\beta}x + D\sinh\sqrt{\beta}x\right)$$

$$w''(x) = \beta\left(-A\sin\sqrt{\beta}x - B\cos\sqrt{\beta}x + C\sinh\sqrt{\beta}x + D\cosh\sqrt{\beta}x\right)$$

$$w'''(x) = \beta\sqrt{\beta}\left(-A\cos\sqrt{\beta}x + B\sin\sqrt{\beta}x + C\cosh\sqrt{\beta}x + D\sinh\sqrt{\beta}x\right)$$

$$w(0) = A + C = 0$$

$$C = -A$$

$$w'(0) = B + D = 0$$

$$D = -B$$

$$w''(L) = \beta\left(-A\sin\sqrt{\beta}L - B\cos\sqrt{\beta}L - A\sinh\sqrt{\beta}L - B\cosh\sqrt{\beta}L\right) = 0$$

$$B = -\frac{\sin\sqrt{\beta}L + \sinh\sqrt{\beta}L}{\cos\sqrt{\beta}L + \cosh\sqrt{\beta}L}A$$

$$\begin{aligned} EIw'''(L) &= EI\beta\sqrt{\beta}\left[-A\left(\cos\sqrt{\beta}L + \cosh\sqrt{\beta}L\right) + B\left(\sin\sqrt{\beta}L - \sinh\sqrt{\beta}L\right)\right] \\ &= k\left[A\left(\sin\sqrt{\beta}L - \sinh\sqrt{\beta}L\right) + B\left(\cos\sqrt{\beta}L - \cosh\sqrt{\beta}L\right)\right] \end{aligned}$$

$$\begin{aligned} &EI\beta\sqrt{\beta}\left[-\left(\cos\sqrt{\beta}L + \cosh\sqrt{\beta}L\right) - \frac{\sin\sqrt{\beta}L + \sinh\sqrt{\beta}L}{\cos\sqrt{\beta}L + \cosh\sqrt{\beta}L}\left(\sin\sqrt{\beta}L - \sinh\sqrt{\beta}L\right)\right] \\ &\quad = k\left[\left(\sin\sqrt{\beta}L - \sinh\sqrt{\beta}L\right) - \frac{\sin\sqrt{\beta}L + \sinh\sqrt{\beta}L}{\cos\sqrt{\beta}L + \cosh\sqrt{\beta}L}\left(\cos\sqrt{\beta}L - \cosh\sqrt{\beta}L\right)\right] \end{aligned}$$

Q4.

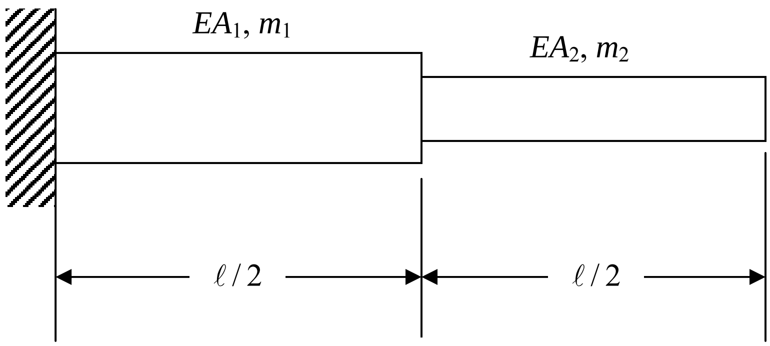


Figure 5

$$EA_1 = 2EA_2$$
$$m_1 = 2m_2$$
$$k_2 = 2\frac{EA_2}{l}$$
$$\left[-\frac{\omega^2}{6}\begin{bmatrix} 2(m_1+m_2) & m_2 \\ m_2 & 2m_2 \end{bmatrix} + \begin{bmatrix} (k_1+k_2) & -k_2 \\ -k_2 & k_2 \end{bmatrix} \right] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$
$$\left[-\frac{\omega^2 m_2}{6}\begin{bmatrix} 2(2+1) & 1 \\ 1 & 2 \end{bmatrix} + \frac{2EA_2}{l}\begin{bmatrix} (2+1) & -1 \\ -1 & 1 \end{bmatrix} \right] \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$
$$\lambda = \frac{\omega^2 m_2}{6}\left(\frac{l}{2EA_2}\right) = \left(\frac{\omega^2 m_2 l}{12EA_2}\right)$$

Let $\therefore \left| -\lambda \begin{bmatrix} 6 & 1 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \right| = 0$

$$\lambda_1 = 0.1640$$
$$\lambda_2 = 1.1088$$

$\omega_1 = 1.4029\sqrt{\frac{EA_2}{m_2 l}}$ and $\omega_2 = 3.6477\sqrt{\frac{EA_2}{m_2 l}}$

Mode shapes:

$(3-6\lambda)\, u_1 = (1+\lambda)\, u_2$ Or: $(1+\lambda)\, u_1 = (1+2\lambda)\, u_2$

Let's consider the second equation:

$$\left(\frac{u_1}{u_2}\right)_i = \frac{1+2\lambda_i}{1+\lambda_i}$$
$$\left(\frac{u_1}{u_2}\right)_1 = 0.5773$$
$$\left(\frac{u_1}{u_2}\right)_2 = -0.5258$$