<u>Problem 1</u>

$$\sum F_1 = -kx_1 + k\left(x_2 - x_1\right) = m\ddot{x}_1$$
$$m\ddot{x}_1 + 2kx_1 - kx_2 = 0$$
$$\sum F_2 = -k\left(x_2 - x_1\right) = m\ddot{x}_2$$
$$m\ddot{x}_2 - kx_1 + kx_2$$
$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$
$$If\ x_1 = X_1 \sin\omega t \quad \Rightarrow \quad \ddot{x}_1 = -\omega^2 X_1 \sin\omega t$$
$$x_2 = X_2 \sin\omega t \quad \Rightarrow \quad \ddot{x}_2 = -\omega^2 X_2 \sin\omega t$$
then
$$-m\omega^2 X_1\left(\sin\omega t\right) + 2kX_1\left(\sin\omega t\right) - kX_2\left(\sin\omega t\right) = 0$$
$$\left(2 - \frac{m\omega^2}{k}\right)X_1 - X_2 = 0 \qquad (a)$$
$$-m\omega^2 X_2\left(\sin\omega t\right) - kX_1\left(\sin\omega t\right) + kX_2\left(\sin\omega t\right) = 0$$
$$X_1 - \left(1 - \frac{m\omega^2}{k}\right)X_2 = 0 \qquad (b)$$

Let $\lambda = \dfrac{m\omega^2}{k}$

$$\begin{bmatrix} (2-\lambda) & -1 \\ 1 & -(1-\lambda) \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$
$$\begin{vmatrix} (2-\lambda) & -1 \\ 1 & -(1-\lambda) \end{vmatrix} = 0$$
$$\lambda^2 - 3\lambda + 1 = 0$$
$$\lambda = \frac{3}{2} \pm \frac{\sqrt{9-4}}{2} = \frac{3}{2} \pm \frac{\sqrt{5}}{2}$$
$$\lambda_1 = \frac{3}{2} - \frac{\sqrt{5}}{2} = 0.382; \quad \Rightarrow \omega_1 = \sqrt{.382\frac{k}{m}}$$
$$\lambda_2 = \frac{3}{2} + \frac{\sqrt{5}}{2} = 2.618; \quad \Rightarrow \omega_1 = \sqrt{2.618\frac{k}{m}}$$

Now substitute ω_1 and ω_2 into (a) or (b)

$$\left(\frac{X_1}{X_2}\right)^{(1)} = \frac{1}{2 - \frac{m\omega^2}{k}}$$

$$\left(\frac{X_1}{X_2}\right)^{(1)} = \frac{1}{2 - \frac{m}{k}\left(.382\frac{k}{m}\right)} = 0.618$$

$$\left(\frac{X_1}{X_2}\right)^{(2)} = \frac{1}{2 - \frac{m}{k}\left(2.618\frac{k}{m}\right)} = -1.618$$

Problem 2

$$\sum M_o = k_2(\theta_2 - \theta_1) - k_1\theta_1 = J_1\ddot{\theta}_1$$

$$J_1\ddot{\theta}_1 + (k_1 + k_2)\theta_1 - k_2\theta_2 = 0 \qquad (a)$$

$$\sum M_{o'} = -k_2(\theta_2 - \theta_1) = J_2\ddot{\theta}_2$$

$$J_2\ddot{\theta}_2 - k_2\theta_1 + k_2\theta_2 = 0 \qquad (b)$$

If $k_1 = k_2$ and $J_1 = J_2$, (a) and (b) become

$$2J_2\ddot{\theta}_1 + 2k_1\theta_1 - k_1\theta_2 = 0 \qquad (c)$$

$$J_2\ddot{\theta}_2 - k_1\theta_1 + k_1\theta_2 = 0 \qquad (d)$$

or

$$J_2\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + k_1\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

If $\theta_1 = A_1 \sin\omega t; \quad \Rightarrow \ddot{\theta}_1 = -\omega^2 A_1 \sin\omega t$

$\theta_2 = A_2 \sin\omega t; \quad \Rightarrow \ddot{\theta}_2 = -\omega^2 A_2 \sin\omega t$

then the matrix equation becomes

$$\begin{bmatrix} -2J_2\omega^2 & 0 \\ 0 & -J_2\omega^2 \end{bmatrix}\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} + \begin{bmatrix} 2k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix}\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$k_1\begin{bmatrix} 2 - \dfrac{2J_2\omega^2}{k_1} & -1 \\ -1 & 1 - \dfrac{J_2\omega^2}{k_1} \end{bmatrix}\begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

If $\lambda = \dfrac{J_2\omega^2}{k_1}$, then

$$k_1 \begin{bmatrix} 2(1-1\lambda) & -1 \\ -1 & (1-\lambda) \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{vmatrix} 2(1-\lambda) & -1 \\ -1 & (1-\lambda) \end{vmatrix} = 0 \qquad (e)$$

$$2(1-\lambda)^2 - 1 = 0$$

$$1-\lambda = \pm\frac{1}{\sqrt{2}}$$

$$\lambda = 1 \pm \frac{1}{\sqrt{2}}$$

$$\lambda_1 = 1 - \frac{1}{\sqrt{2}} = 0.293; \quad \Rightarrow \omega_1 = \sqrt{0.293\left(\frac{k_1}{J_2}\right)}$$

$$\lambda_2 = 1 + \frac{1}{\sqrt{2}} = 1.707; \quad \Rightarrow \omega_1 = \sqrt{1.707\left(\frac{k_1}{J_2}\right)}$$

From (e)

$$2(1-\lambda)A_1 - A_2 = 0$$

$$\left(\frac{A_1}{A_2}\right) = \frac{1}{2(1-\lambda)}$$

$$\left(\frac{A_1}{A_2}\right)^{(1)} = \frac{1}{2(1-\lambda_1)} = \frac{1}{2(1-0.293)} = 0.707$$

$$\left(\frac{A_1}{A_2}\right)^{(2)} = \frac{1}{2(1-\lambda_2)} = \frac{1}{2(1-1.707)} = -0.707$$

Problem 3

Lower mass:

$$\sum F_y = -T_2 \cos\theta_2 + mg = 0$$

$$T_2 \cos\theta_2 = mg$$

$$for\ small\ \theta_2, T_2 \approx mg$$

$$\sum F_x = -T_2 \sin\theta_2 \approx -mg\theta_2 = ml\left(\ddot{\theta}_1 + \ddot{\theta}_2\right)$$

$$\ddot{\theta}_1 + \ddot{\theta}_2 + \frac{g}{l} = 0 \qquad (a)$$

Upper mass:

$$\sum F_y = -T_1 \cos\theta_1 + T_2 \cos\theta_2 + mg = 0$$

$$T_1 \cos\theta_1 = 2mg$$

$$for\ small\ \theta_1, T_1 \approx 2mg$$

$$\sum F_x = -T_1 \sin\theta_1 + T_2 \sin\theta_2 \approx -2mg\theta_1 + mg\theta_2 = ml\ddot{\theta}_1$$

$$\ddot{\theta}_1 + 2\frac{g}{l}\theta_1 - \frac{g}{l}\theta_2 = 0 \qquad (b)$$

If $\theta_1 = A_1 \sin\omega t; \quad \Rightarrow \ddot{\theta}_1 = -\omega^2 A_1 \sin\omega t$

$\theta_2 = A_2 \sin\omega t; \quad \Rightarrow \ddot{\theta}_2 = -\omega^2 A_2 \sin\omega t$

then (b) and (a) become

$$-A_1\omega^2\left(\sin\omega t\right) + 2A_1\left(\frac{g}{l}\right)\left(\sin\omega t\right) - A_2\left(\frac{g}{l}\right)\left(\sin\omega t\right) = 0$$

$$A_1\left(2\frac{g}{l} - \omega^2\right) - A_2\left(\frac{g}{l}\right) = 0 \qquad (c)$$

and

$$-A_1\omega^2\left(\sin\omega t\right) - A_2\omega^2\left(\sin\omega t\right) + A_2\left(\frac{g}{l}\right)\left(\sin\omega t\right) = 0$$

$$-A_1\omega^2 + A_2\left(\frac{g}{l} - \omega^2\right) = 0 \qquad (d)$$

If we multiply (c) and (d) by $\left(\frac{l}{g}\right)$ and make the substitution $\lambda = \frac{l\omega^2}{g}$,

the result is

$$(2-\lambda)A_1 - A_2 = 0 \qquad (e)$$

$$-\lambda A_1 + (1-\lambda)A_2 = 0 \qquad (f)$$

$$\begin{bmatrix} (2-\lambda) & -1 \\ -1 & (1-\lambda) \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{vmatrix} (2-\lambda) & -1 \\ -\lambda & (1-\lambda) \end{vmatrix} = 0$$

$$(2-\lambda)(1-\lambda) - \lambda = 0$$

$$\lambda^2 - 4\lambda + 2 = 0$$

$$\lambda = 2 \pm \frac{\sqrt{16-8}}{2} = 2 \pm \sqrt{2}$$

$$\lambda_1 = 2 - \sqrt{2} = 0.586; \quad \Rightarrow \omega_1 = \sqrt{.586\frac{g}{l}}$$

$$\lambda_2 = 2 + \sqrt{2} = 3.414; \quad \Rightarrow \omega_2 = \sqrt{3.414\frac{g}{l}}$$

Substituting these values into either of (e) or (f) yields

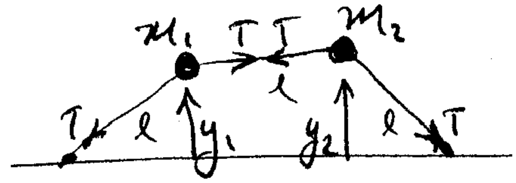
$$(2-\lambda)A_1 - A_2 = 0$$

$$\left(\frac{A_1}{A_2}\right) = \frac{1}{2-\lambda}$$

$$\left(\frac{A_1}{A_2}\right)^{(1)} = \frac{1}{2-(0.586)} = 0.707$$

$$\left(\frac{A_1}{A_2}\right)^{(2)} = \frac{1}{2-(3.414)} = -0.707$$

Problem 4



$$\sum F_{y_1} = T\left(\frac{y_2 - y_1}{L}\right) - T\left(\frac{y_1}{L}\right) = m_1\ddot{y}_1$$

$$m_1\ddot{y}_1 + 2T\left(\frac{y_1}{L}\right) - T\left(\frac{y_2}{L}\right) = 0$$

$$\frac{m_1 L}{T}\ddot{y}_1 + 2y_1 - y_2 = 0 \qquad (a)$$

$$\sum F_{y_2} = -T\left(\frac{y_2 - y_1}{L}\right) - T\left(\frac{y_2}{L}\right) = m_2\ddot{y}_2$$

$$m_2\ddot{y}_2 - T\left(\frac{y_1}{L}\right) + 2T\left(\frac{y_2}{L}\right) = 0$$

$$\frac{m_2 L}{T}\ddot{y}_2 - y_1 + 2y_2 = 0 \qquad (b)$$

$$\begin{bmatrix} \left(\frac{m_1 L}{T}\right) & 0 \\ 0 & \left(\frac{m_2 L}{T}\right) \end{bmatrix} \begin{Bmatrix} \ddot{y}_1 \\ \ddot{y}_1 \end{Bmatrix} + \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Problem 5

In Problem 4, if $m_1 = m_2 = m$, then

$$\left(\frac{mL}{T}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{Bmatrix} + \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (a)$$

Let $y_i = Y_i \sin\omega t;\ \ddot{y}_i = -\omega^2 Y_i \sin\omega t$. Eq (a) becomes

$$\left(-\frac{mL\omega^2}{T}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} Y_1 \\ Y_2 \end{Bmatrix} (\sin\omega t) + \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{Bmatrix} Y_1 \\ Y_2 \end{Bmatrix} (\sin\omega t) = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Now let $\lambda = \frac{mL\omega^2}{T}$, so that this equation becomes

$$\begin{bmatrix} (2-\lambda) & -1 \\ -1 & (2-\lambda) \end{bmatrix} \begin{Bmatrix} Y_1 \\ Y_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (a)$$