Problem 1

Left Pendulum:

$$\sum M_o = k(\theta_2 - \theta_1) - mgl\sin\theta_1 = ml^2\ddot{\theta}_1$$

$$ml^2\ddot{\theta}_1 + (k + mgl)\theta_1 - k\theta_2 = 0 \qquad (a)$$

Right Pendulum:

$$\sum M_{o'} = -k(\theta_2 - \theta_1) - mgl\sin\theta_2 = ml^2\ddot{\theta}_2$$

$$ml^2\ddot{\theta}_2 - k\theta_1 + (k + mgl)\theta_2 = 0 \qquad (b)$$

If $\theta_1 = A_1 \sin\omega t; \quad \Rightarrow \ddot{\theta}_1 = -\omega^2 A_1 \sin\omega t$

$\theta_2 = A_2 \sin\omega t; \quad \Rightarrow \ddot{\theta}_2 = -\omega^2 A_2 \sin\omega t$

equations (a) and (b) become

$$-ml^2\omega^2 A_1(\sin\omega t) + (k + mgl)A_1(\sin\omega t) - kA_2(\sin\omega t) = 0$$

$$(k + mgl - ml^2\omega^2)A_1 - kA_2 = 0 \qquad (c)$$

and

$$-ml^2\omega^2 A_2(\sin\omega t) + (k + mgl)A_2(\sin\omega t) - kA_1(\sin\omega t) = 0$$

$$-kA_1 + (k + mgl - ml^2\omega^2)A_2 = 0 \qquad (d)$$

Now divide both equations by ml^2 and these become

$$\left(\frac{k}{ml^2} + \frac{g}{l} - \omega^2\right)A_1 - \frac{k}{ml^2}A_2 = 0 \qquad (e)$$

$$-\frac{k}{ml^2}A_1 + \left(\frac{k}{ml^2} + \frac{g}{l} - \omega^2\right)A_2 = 0 \qquad (f)$$

$$\begin{bmatrix} \left(\frac{k}{ml^2} + \frac{g}{l} - \omega^2\right) & -\frac{k}{ml^2} \\ -\frac{k}{ml^2} & \left(\frac{k}{ml^2} + \frac{g}{l} - \omega^2\right) \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{vmatrix} \left(\frac{k}{ml^2} + \frac{g}{l} - \omega^2\right) & -\frac{k}{ml^2} \\ -\frac{k}{ml^2} & \left(\frac{k}{ml^2} + \frac{g}{l} - \omega^2\right) \end{vmatrix} = 0 \qquad (g)$$

Finding the roots of (g) produces

$$\left(\frac{k}{ml^2}+\frac{g}{l}-\omega^2\right)^2=\left(\frac{k}{ml^2}\right)^2$$

$$\left(\frac{k}{ml^2}+\frac{g}{l}-\omega^2\right)=\pm\frac{k}{ml^2}$$

$$\omega^2=\frac{k}{ml^2}+\frac{g}{l}\pm\frac{k}{ml^2}$$

$$\omega=\sqrt{\frac{k}{ml^2}+\frac{g}{l}\pm\frac{k}{ml^2}}$$

$$\omega_1=\sqrt{\frac{k}{ml^2}+\frac{g}{l}-\frac{k}{ml^2}}=\sqrt{\frac{g}{l}}$$

$$\omega_2=\sqrt{\frac{k}{ml^2}+\frac{g}{l}+\frac{k}{ml^2}}=\sqrt{\frac{g}{l}+\frac{2k}{ml^2}}$$

From equation (e)

$$\left(\frac{A_1}{A_2}\right)=\frac{\frac{k}{ml^2}}{\frac{k}{ml^2}+\frac{g}{l}-\omega^2}$$

$$\left(\frac{A_1}{A_2}\right)^{(1)}=\frac{\frac{k}{ml^2}}{\frac{k}{ml^2}+\frac{g}{l}-\frac{g}{l}}=1$$

$$\left(\frac{A_1}{A_2}\right)^{(2)}=\frac{\frac{k}{ml^2}}{\frac{k}{ml^2}+\frac{g}{l}-\frac{g}{l}-\frac{2k}{ml^2}}=-1$$

Now we write for the general case,

$$\begin{Bmatrix}\theta_1\\ \theta_2\end{Bmatrix}=\begin{Bmatrix}1\\ 1\end{Bmatrix}A_1\sin(\omega_1 t+\psi_1)+\begin{Bmatrix}-1\\ 1\end{Bmatrix}A_2\sin(\omega_2 t+\psi_2)\qquad (h)$$

$$\theta_1=A_1\sin(\omega_1 t+\psi_1)-A_2\sin(\omega_2 t+\psi_2)\qquad (i)$$

$$\theta_2=A_1\sin(\omega_1 t+\psi_1)+A_2\sin(\omega_2 t+\psi_2)\qquad (j)$$

so

$$\dot{\theta}_1=A_1\omega_1\cos(\omega_1 t+\psi_1)-A_2\omega_2\cos(\omega_2 t+\psi_2)\qquad (k)$$

$$\dot{\theta}_2=A_1\omega_1\cos(\omega_1 t+\psi_1)+A_2\omega_2\cos(\omega_2 t+\psi_2)\qquad (l)$$

Apply the initial velocity conditions, $\dot{\theta}(0)=\dot{\theta}(0)=0$ to (k) and (l)

$$A_1\omega_1\cos\psi_1 = A_2\omega_2\cos\psi_2$$
and
$$A_1\omega_1\cos\psi_1 = -A_2\omega_2\cos\psi_2$$
$$2A_1\omega_1\cos\psi_1 = 0$$
$$\therefore \psi_1 = \psi_2 = 90°$$

which means that equations (i) and (j) become

$$\theta_1 = A_1\sin(\omega_1 t + 90) - A_2\sin(\omega_2 t + 90)$$
$$\theta_2 = A_1\sin(\omega_1 t + 90) + A_2\sin(\omega_2 t + 90)$$
or
$$\theta_1 = A_1\cos(\omega_1 t) - A_2\cos(\omega_2 t) \qquad (m)$$
$$\theta_2 = A_1\cos(\omega_1 t) + A_2\cos(\omega_2 t) \qquad (n)$$

Now apply the initial displacement conditions, t = 0, $\theta_1 = 0$, $\theta_2 = \theta_0$, so that

$$0 = A_1 - A_2$$
$$\theta_0 = A_1 + A_2$$
$$2A_1 = \theta_0$$
$$A_1 = \frac{\theta_0}{2}$$
$$A_2 = \frac{\theta_0}{2}$$
$$\theta_1 = \frac{\theta_0}{2}\left[\cos(\omega_1 t) - \cos(\omega_2 t)\right] \qquad (o)$$
$$\theta_2 = \frac{\theta_0}{2}\left[\cos(\omega_1 t) + \cos(\omega_2 t)\right] \qquad (p)$$

Equations (o) and (p) are of the same form as the Example 5.2.2 for which the beat period was given as $\tau_b = \dfrac{2\pi}{\omega_1 - \omega_1}$. Then

$$\omega_1 = \sqrt{\frac{g}{l}} = \sqrt{\frac{386}{19.3}} = \sqrt{20} = 4.4721$$
$$\omega_2 = \sqrt{\frac{g}{l} + \frac{2k}{ml^2}} = \sqrt{20 + \frac{2\left(20/12\right)}{\left(\frac{3.86}{32.174}\right)\left(\frac{19.3}{12}\right)^2}} = \sqrt{30.7385} = 5.5442$$

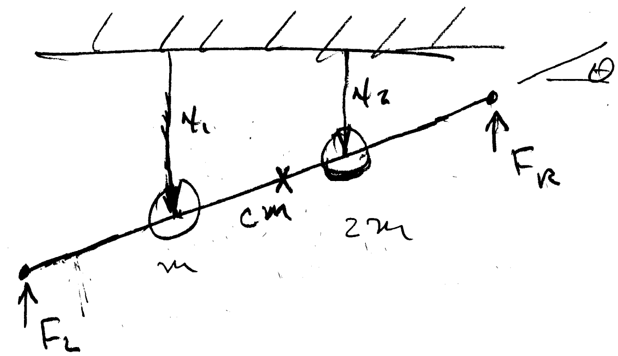
so

$$\tau_b = \frac{2\pi}{\omega_2 - \omega_1} = \frac{2\pi}{5.5442 - 4.4721}$$

$$\tau_b = 5.8606\ s$$

Problem 2

To find the location of the center of mass (cm)along the massless rod, let q be the distance of cm from mass m. Then L-q will be the distance from cm to mass 2. If we take moments (static) about cm, then



-m*q + 2m(L-q) = 0
3*m*q = 2*m*L
q = 2L/3

The cm is thus located 2L/3 from mass m.

The angle θ is used for calculational purposes and is defined (for small θ by $\theta = \frac{x_1 - x_2}{L}$. The displacement of the left end of the rod is thus given by $x_L = x_1 + L\left(\frac{x_1 - x_2}{L}\right) = 2x_1 - x_2$, so the force of the left spring is $F_L = k\left(2x_1 - x_2\right)$. Similarly, the displacement of the right end of the rod is $x_R = x_2 - L\left(\frac{x_1 - x_2}{L}\right) = 2x_2 - x_1$, so the force of the spring on the right end is $F_R = k\left(2x_2 - x_1\right)$.

Now that all the forces are defined take the sum of forces and sum of moments to produce the equations of motion.

$$\sum F_x = -k\left(2x_1 - x_2\right) - k\left(2x_2 - x_1\right) = m\ddot{x}_1 + 2m\ddot{x}_2$$

$$m\ddot{x}_1 + 2m\ddot{x}_2 + kx_1 + kx_2 = 0 \qquad (a)$$

The inertia of the masses about cm is given by

$$J_{cm} = m\left(\frac{2L}{3}\right)^2 + 2m\left(\frac{L}{3}\right)^2 = \frac{2mL^2}{3}$$, so

$$\sum M_{cm} = k(2x_2 - x_1)\left(\frac{4L}{3}\right) - k(2x_1 - x_2)\left(\frac{5L}{3}\right) = J_{cm}\ddot{\theta} = \frac{2mL^2}{3}\left(\frac{\ddot{x}_1 - \ddot{x}_2}{L}\right)$$

$$\frac{2mL^2}{3}\left(\frac{\ddot{x}_1 - \ddot{x}_2}{L}\right) + \frac{14kL}{3}x_1 - \frac{13kL}{3}x_2 = 0$$

$$2m(\ddot{x}_1 - \ddot{x}_2) + 14kx_1 - 13kx_2 = 0 \qquad (b)$$

Equations (a) and (b) can be put into matrix form as

$$m\begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + k\begin{bmatrix} 1 & 1 \\ 14 & -13 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \{0\} \qquad (c)$$

If $x_i = X_i \sin\omega t \Rightarrow \ddot{x}_i = -\omega^2 X_i \sin\omega t = -\omega^2 x_i$

$$\begin{bmatrix} 1 - \frac{m\omega^2}{k} & 1 - 2\frac{m\omega^2}{k} \\ 14 - 2\frac{m\omega^2}{k} & -\left(13 - 2\frac{m\omega^2}{k}\right) \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \{0\}$$

$$If\, \lambda = \frac{m\omega^2}{k},$$

$$\begin{bmatrix} (1-\lambda) & (1-2\lambda) \\ (14-2\lambda) & -(13-2\lambda) \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \{0\} \qquad (d)$$

$$\begin{vmatrix} (1-\lambda) & (1-2\lambda) \\ (14-2\lambda) & -(13-2\lambda) \end{vmatrix} = 0$$

$$-(1-\lambda)(13-2\lambda) - (14-2\lambda)(1-2\lambda) = 0$$

$$13 - 15\lambda + 2\lambda^2 + 14 - 30\lambda + 4\lambda^2 = 0$$

$$6\lambda^2 - 45\lambda + 27 = 0$$

$$2\lambda^2 - 15\lambda + 9 = 0$$

and

$$\lambda = \frac{15}{4} \pm \frac{1}{4}\sqrt{225-72} = \frac{15}{4} \pm \frac{1}{4}\sqrt{153}$$

$$\lambda_1 = \frac{15}{4} - \frac{1}{4}\sqrt{153} = 0.6577 \qquad \omega_1 = .811\sqrt{\frac{k}{m}}$$

$$\lambda_2 = \frac{15}{4} + \frac{1}{4}\sqrt{153} = 6.8423 \qquad \omega_2 = 2.616\sqrt{\frac{k}{m}}$$

From the first motion equation contained in (d)

$$(1-\lambda)x_1 + (1-2\lambda)x_2 = 0$$

$$\left(\frac{x_1}{x_2}\right) = -\frac{1-2\lambda}{1-\lambda}$$

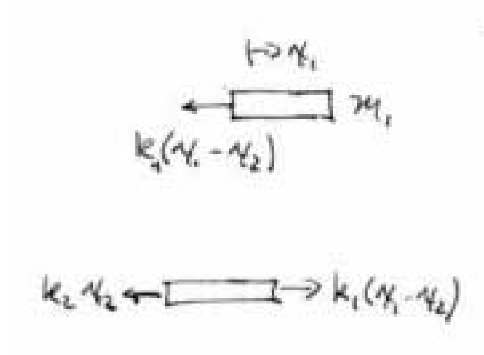
$$\left(\frac{x_1}{x_2}\right)^{(1)} = -\frac{1-2\lambda_1}{1-\lambda_1} = -\frac{1-2(.6577)}{1-(.6577)} = 0.9214$$

$$\left(\frac{x_1}{x_2}\right)^{(2)} = -\frac{1-2\lambda_2}{1-\lambda_2} = -\frac{1-2(6.8426)}{1-(6.8423)} = -2.1712$$

Thus,

$$\phi_1 = \begin{Bmatrix} .9214 \\ 1 \end{Bmatrix} \qquad \phi_2 = \begin{Bmatrix} -2.1712 \\ 1 \end{Bmatrix}$$

Problem 3



The free body diagrams for the two masses are shown at right. For mass 1

$$\sum F_{x,1} = -k_1\left(x_1 - x_2\right) = m_1\ddot{x}_1$$

$$m_1\ddot{x}_1 + k_1x_1 - k_1x_2 = 0 \qquad (a)$$

while for mass 2

$$\sum F_{x,2} = -k_2x_2 + k_1\left(x_1 - x_2\right) = m_2\ddot{x}_2$$

$$m2\ddot{x}_2 - k_1x_1 + \left(k_1 + k_2\right)x_2 = 0 \qquad (b)$$

Let $x_i = X_i \sin\omega t; \quad \ddot{x}_i = -\omega^2 X_i \sin\omega t$. Then

$$-m_1\omega^2 X_1\left(\cancel{\sin\omega t}\right) + k_1X_1\left(\cancel{\sin\omega t}\right) - k_1X_2\left(\cancel{\sin\omega t}\right) = 0$$

$$\left(k_1 - m_1\omega^2\right)X_1 - k_1X_2 = 0$$

$$\left(1 - \frac{m_1\omega^2}{k_1}\right)X_1 - X_2 = 0 \qquad (c)$$

$$-m_2\omega^2 X_2\left(\cancel{\sin\omega t}\right) - k_1X_1\left(\cancel{\sin\omega t}\right) + \left(k_1 + k_2\right)X_2\left(\cancel{\sin\omega t}\right) = 0$$

$$-k_1X_1 + \left(k_1 + k_2 - m_2\omega^2\right)X_2 = 0$$

$$-X_1 + \left(\frac{k_1 + k_2}{k_1} - \frac{m_2\omega^2}{k_1}\right)X_2 = 0 \qquad (d)$$

Now substitute $m_2 = 2m_1$, $k_2 = 2k_1$, and $\lambda = \frac{m_1\omega^2}{k_1}$ into eqns (c) and (d) and put in matrix form

$$\begin{bmatrix} (1-\lambda) & -1 \\ -1 & (3-2\lambda) \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \{0\} \qquad (e)$$

from which we get the characteristic equation

$$\begin{vmatrix} (1-\lambda) & -1 \\ -1 & (3-2\lambda) \end{vmatrix} = 0$$

$$3 - 5\lambda + 2\lambda^2 - 1 = 0$$

$$2\lambda^2 - 5\lambda + 2 = 0$$

from which we get the eigenvalues

$$\lambda = \frac{5}{4} \pm \frac{1}{4}\sqrt{25-16} = \frac{5}{4} \pm \frac{3}{4}$$

$$\lambda_1 = \frac{5}{4} - \frac{3}{4} = \frac{1}{2}$$

$$\lambda_2 = \frac{5}{4} + \frac{3}{4} = 2$$

From the second equation of motion we get

$$-X_1 + (3-2\lambda) X_2 = 0$$

$$\left(\frac{X_1}{X_2}\right) = (3-2\lambda)$$

$$\left(\frac{X_1}{X_2}\right)^{(1)} = (3-2\lambda_1) = 2$$

$$\left(\frac{X_1}{X_2}\right)^{(2)} = (3-2\lambda_2) = -1$$

Problem 5

$$\frac{X_1 k_1}{F_0} = \frac{\left[1-\left(\frac{\omega}{\omega_{22}}\right)^2\right]}{\left[1+\frac{k_2}{k_1}-\left(\frac{\omega}{\omega_{11}}\right)^2\right]\left[1-\left(\frac{\omega}{\omega_{22}}\right)^2\right]-\frac{k_2}{k_1}}$$

where

$$\omega_{11}^2 = \frac{k_1}{m_1} \ \ and \ \omega_{22}^2 = \frac{k_2}{m_2}$$

If we select k_2 so that $\omega_{22} = \omega$, then $X_1 = 0$ and the primary mass has no oscillation. Thus,

$$k_2 = m_2\omega_{22}^2 = m_2\omega^2$$

$$k_2 = \left(\frac{50}{32.174}\right)\left(\frac{2\pi\times 1800}{60}\right)^2$$

$$k_2 = 55216.3\ N/ft = 4601.4\ N/in.$$

$$F_0 = (me)\omega^2 = k_2 X_2$$

$$X_2 = \frac{(me)\omega^2}{k_2} = \frac{(me)\omega^2}{M_2\omega^2}$$

$$X_2 = \frac{(me)}{M_2} = \frac{2\ lb-in}{50\ lb}$$

$$X_2 = 0.040\ in.$$
