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Mechanical Vibrations
MENG470
Homework Assignment 8

Q.1

Recognize: 1 DOF therefore $q = x$

Using Lagranges Equation:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} + \frac{\partial RD}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = Q_i \qquad for \quad i = 1$$

where L is called the Lagrangian:

$$L = KE - PE$$

Kinetic energy:

$$KE = \frac{1}{2} m v^2 = \frac{1}{2} m \dot{x}^2$$

Potential energy

$$PE = \frac{1}{2} k_1 (x - y_1)^2 + \frac{1}{2} k_2 (y_2 - x)^2$$

Lagrangian:

$$L = KE - PE$$
$$= \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k_1 (x - y_1)^2 - \frac{1}{2} k_2 (y_2 - x)^2$$

Rayleigh Dissipation:

$$RD = \frac{1}{2} c (\dot{y}_2 - \dot{x})^2$$

Generalized Forces:

None to deal with: Q=0

Lagrange's Equation:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} + \frac{\partial RD}{\partial \dot{q}} - \frac{\partial L}{\partial q} = Q \qquad \rightarrow \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} + \frac{\partial RD}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0$$

Partial Derivative with respect to displacement, $\frac{\partial L}{\partial x}$

$$\frac{\partial L}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{2} m \dot{x}^2 - \frac{1}{2} k_1 (x - y_1)^2 - \frac{1}{2} k_2 (y_2 - x)^2\right)$$
$$= -k_1 (x - y_1) + k_2 (y_2 - x)$$
$$= -(k_1 + k_2) x + k_1 y_1 + k_2 y_2$$

Partial Derivative with respect to velocity, $\frac{\partial L}{\partial \dot{x}}$

$$\frac{\partial L}{\partial \dot{x}} = \frac{\partial}{\partial \dot{x}}\left(\frac{1}{2} m \dot{x}^2 - \frac{1}{2} k_1 (x - y_1)^2 - \frac{1}{2} k_2 (y_2 - x)^2 \right)$$
$$= m \dot{x}$$

Time Derivative of the Partial with respect to velocity, $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}}$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{d}{dt}\left(m\dot{x} \right)$$
$$= m \ddot{x}$$

Partial Derivative of the Raleigh Dissipation with respect to velocity,

$$\frac{\partial RD}{\partial \dot{x}} = \frac{\partial}{\partial \dot{x}}\left(\frac{1}{2} c(\dot{y}_2 - \dot{x})^2 \right)$$
$$= -c(\dot{y}_2 - \dot{x})$$
$$= -c\dot{y}_2 + c\dot{x}$$

Then the Lagrange Equation can be written:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} + \frac{\partial RD}{\partial \dot{x}} - \frac{\partial L}{\partial x} = Q$$

$$\left\{ m\ddot{x} \right\} + \left\{ c\dot{x} - c\dot{y}_2 \right\} - \left\{ -(k_1 + k_2)x + k_1 y_1 + k_2 y_2 \right\} = 0$$

Finally simplify the EOM:

$$m\ddot{x} + c\dot{x} + (k_1 + k_2)x_1 - c\dot{y}_2 - k_1 y_1 - k_2 y_2 = 0$$

for k_1 = 50 N/m, k_2=20 N/m, c = 2 N-s/m, and m = 10 kg.

$$10\ddot{x} + 2\dot{x} + (50 + 20)x_1 - 2\dot{y}_2 - 50y_1 - 20y_2 = 0$$
$$10\ddot{x} + 2\dot{x} + 70x_1 - 2\dot{y}_2 - 50y_1 - 20y_2 = 0$$
$$\ddot{x} + \frac{1}{5}\dot{x} + 7x_1 - \frac{1}{5}\dot{y}_2 - 5y_1 - 2y_2 = 0$$
$$\ddot{x} + \frac{1}{5}\dot{x} + 7x = \frac{1}{5}\dot{y}_2 + 5y_1 + 2y_2$$

Q2.

Generalized Variables

$q_1 = x_1$

$q_2 = x_2$

where L is called the Lagrangian:

$L = KE - PE$

Using Lagrange's Equation:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} + \frac{\partial RD}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = Q_i \qquad for \quad i = 1,2$$

Kinetic Energy:

$$KE = \frac{1}{2}m_1 v_1^{\,2} + \frac{1}{2}m_2 v_2^{\,2} = \frac{1}{2}m_1 \dot{x}_1^{\,2} + \frac{1}{2}m_2 \dot{x}_2^{\,2}$$

Potential Energy

$$PE = \frac{1}{2}k_1(x_2 - x_1)^2 + \frac{1}{2}k_2 x_2^{\,2}$$

Lagrangian:

$$L = KE - PE$$

$$L = \left\{\frac{1}{2}m_1\dot{x}_1^{\,2} + \frac{1}{2}m_2\dot{x}_2^{\,2}\right\} - \left\{\frac{1}{2}k_1(x_2 - x_1)^2 + \frac{1}{2}k_2 x_2^{\,2}\right\}$$

$$= \frac{1}{2}m_1\dot{x}_1^{\,2} + \frac{1}{2}m_2\dot{x}_2^{\,2} - \frac{1}{2}k_1(x_2 - x_1)^2 - \frac{1}{2}k_2 x_2^{\,2}$$

Rayleigh Dissipation Factor:

$$RD = \frac{1}{2}c_1(v_2 - v_1)^2 + \frac{1}{2}c_2 v_2^{\,2} = \frac{1}{2}c_1(\dot{x}_2 - \dot{x}_1)^2 + \frac{1}{2}c_2\dot{x}_2^{\,2}$$

Generalized Forces, Q_1 and Q_2 :

No external forces given: $Q_1 = Q_2 = 0$

Lagrange's Equations: Use

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} + \frac{\partial RD}{\partial \dot{x}_1} - \frac{\partial L}{\partial x_1} = Q_1 \quad \text{and} \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_2} + \frac{\partial RD}{\partial \dot{x}_2} - \frac{\partial L}{\partial x_2} = Q_2$$

Partial Derivative with respect to displacement, $\frac{\partial L}{\partial x_1}$ and $\frac{\partial L}{\partial x_2}$

$$\frac{\partial L}{\partial x_1} = \frac{\partial}{\partial x_1}\left(\frac{1}{2}m_1\dot{x}_1^{\,2} + \frac{1}{2}m_2\dot{x}_2^{\,2} - \frac{1}{2}k_1(x_2 - x_1)^2 - \frac{1}{2}k_2 x_2^{\,2}\right)$$

$$= k_1(x_2 - x_1)$$
$$= k_1x_2 - k_1x_1$$
$$\frac{\partial L}{\partial x_2} = \frac{\partial}{\partial x_2}\left(\frac{1}{2}m_1\dot{x}_1{}^2 + \frac{1}{2}m_2\dot{x}_2{}^2 - \frac{1}{2}k_1(x_2 - x_1)^2 - \frac{1}{2}k_2x_2{}^2\right)$$
$$= -k_1(x_2 - x_1) - k_2x_2$$
$$= -(k_1 + k_2)x_2 + k_1x_1$$

Partial Derivative with respect to velocity, $\frac{\partial L}{\partial \dot{x}_1}$ and $\frac{\partial L}{\partial \dot{x}_2}$

$$\frac{\partial L}{\partial \dot{x}_1} = \frac{\partial}{\partial \dot{x}_1}\left(\frac{1}{2}m_1\dot{x}_1{}^2 + \frac{1}{2}m_2\dot{x}_2{}^2 - \frac{1}{2}k_1(x_2 - x_1)^2 - \frac{1}{2}k_2x_2{}^2\right)$$
$$= m_1\dot{x}_1$$

$$\frac{\partial L}{\partial \dot{x}_2} = \frac{\partial}{\partial \dot{x}_2}\left(\frac{1}{2}m_1\dot{x}_1{}^2 + \frac{1}{2}m_2\dot{x}_2{}^2 - \frac{1}{2}k_1(x_2 - x_1)^2 - \frac{1}{2}k_2x_2{}^2\right)$$
$$= m_2\dot{x}_2$$

Time Derivative of the Partial with respect to velocity, $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1}$ and $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_2}$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} = \frac{d}{dt}\left(m_1\dot{x}_1\right) = m_1\ddot{x}_1$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_2} = \frac{d}{dt}\left(m_2\dot{x}_2\right) = m_2\ddot{x}_2$$

Partial Derivative of the Raleigh Dissipation with respect to velocity,

$$\frac{\partial RD}{\partial \dot{x}_1} = \frac{\partial}{\partial \dot{x}_1}\left(\frac{1}{2}c_1(\dot{x}_2 - \dot{x}_1)^2 + \frac{1}{2}c_2\dot{x}_2{}^2\right)$$
$$= -c_1(\dot{x}_2 - \dot{x}_1)$$
$$= -c_1\dot{x}_2 + c_1\dot{x}_1$$

$$\frac{\partial RD}{\partial \dot{x}_2} = \frac{\partial}{\partial \dot{x}_2}\left(\frac{1}{2}c_1(\dot{x}_2 - \dot{x}_1)^2 + \frac{1}{2}c_2\dot{x}_2{}^2\right)$$
$$= c_1(\dot{x}_2 - \dot{x}_1) + c_2\dot{x}_2$$

$$= (c_1 + c_2)\dot{x}_2 - c_1\dot{x}_1$$

Then the Lagrange Equations can be written:

for x_1:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} + \frac{\partial RD}{\partial \dot{x}_1} - \frac{\partial L}{\partial x_1} = Q_1$$

$$\left\{m_1\ddot{x}_1\right\} + \left\{-c_1\dot{x}_2 + c_1\dot{x}_1\right\} - \left\{k_1x_2 - k_1x_1\right\} = 0$$
$$m_1\ddot{x}_1 + c_1\dot{x}_1 + k_1x_1 - c_1\dot{x}_2 - k_1x_2 = 0$$

and for x_2:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_2} + \frac{\partial RD}{\partial \dot{x}_2} - \frac{\partial L}{\partial x_2} = Q_2$$

$$\left\{m_2\ddot{x}_2\right\} + \left\{(c_1 + c_2)\dot{x}_2 - c_1\dot{x}_1\right\} - \left\{-(k_1 + k_2)x_2 + k_1x_1\right\} = 0$$
$$m_2\ddot{x}_2 + (c_1 + c_2)\dot{x}_2 + (k_1 + k_2)x_2 - c_1\dot{x}_1 - k_1x_1 = 0$$

Finally simplify the EOMs:

$$m_1\ddot{x}_1 + c_1\dot{x}_1 + k_1x_1 - c_1\dot{x}_2 - k_1x_2 = 0$$

and

$$m_2\ddot{x}_2 + (c_1 + c_2)\dot{x}_2 + (k_1 + k_2)x_2 - c_1\dot{x}_1 - k_1x_1 = 0$$

Q3.

Recognize: 1 DOF therefore $q = \theta$

Using Lagranges Equation:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} + \frac{\partial RD}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = Q_i \qquad for \quad i = 1$$

where *L* is called the Lagrangian:

$$L = KE - PE$$

First describe some of the kinematics:

Velocity of the bob is due to both the rotation and also due to the velocity of the base.

$$\vec{v}_B = \vec{v}_A + \vec{v}_{B/A}$$

$$\vec{v}_B = \dot{y}\,\hat{j} + \omega L\cos\theta\,\hat{i} + \omega L\sin\theta\,\hat{j}$$

$$\vec{v}_B = \left(\omega L\cos\theta\right)\hat{i} + \left(\dot{y} + \omega L\sin\theta\right)\hat{j}$$

therefore the magnitude of the velocity is given as

$$v = |\vec{v}_B| = \sqrt{\left(\omega L\cos\theta\right)^2 + \left(\dot{y} + \omega L\sin\theta\right)^2}$$

Kinetic energy:

$$KE = \frac{1}{2}mv^2$$

$$= \frac{1}{2}m\left[\left(\omega L\cos\theta\right)^2 + \left(\dot{y} + \omega L\sin\theta\right)^2\right]$$

$$= \frac{1}{2}m\left(\omega L\cos\theta\right)^2 + \frac{1}{2}m\left(\dot{y} + \omega L\sin\theta\right)^2$$

$$= \frac{1}{2}m\left(\dot{\theta}L\cos\theta\right)^2 + \frac{1}{2}m\left(\dot{y} + \dot{\theta}L\sin\theta\right)^2$$

$$= \frac{1}{2}m\left(\dot{\theta}L\cos\theta\right)^2 + \frac{1}{2}m\left(\dot{y}^2 + 2\dot{y}\dot{\theta}L\sin\theta + \dot{\theta}^2L^2\sin^2\theta\right)$$

$$= \frac{1}{2}m\dot{\theta}^2L^2\cos\theta^2 + \frac{1}{2}m\dot{y}^2 + m\dot{y}\dot{\theta}L\sin\theta + \frac{1}{2}m\dot{\theta}^2L^2\sin^2\theta$$

$$= \frac{1}{2}m\dot{\theta}^2L^2 + \frac{1}{2}m\dot{y}^2 + m\dot{y}\dot{\theta}L\sin\theta$$

Potential energy

$$PE = mgh \qquad \text{where} \quad h = y + (L - L\cos\theta)$$

so

$$PE = mgy + mg(L - L\cos\theta)$$

Lagrangian:

$$L = KE - PE$$
$$= \frac{1}{2} m \left(\dot{\theta} L \cos\theta \right)^2 + \frac{1}{2} m \left(\dot{y} + \dot{\theta} L \sin\theta \right)^2 - mgy - mg(L - L\cos\theta)$$

Rayleigh Dissipation:
No damping: RD = 0

Generalized Forces:
None to deal with: Q=0

Lagrange's Equation:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} + \frac{\partial RD}{\partial \dot{q}} - \frac{\partial L}{\partial q} = Q \quad \rightarrow \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} + \frac{\partial RD}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = Q$$

Partial Derivative with respect to displacement, $\frac{\partial L}{\partial \theta}$

$$\frac{\partial L}{\partial \theta} = \frac{\partial}{\partial \theta}\left(\frac{1}{2} m \left(\dot{\theta} L \cos\theta \right)^2 + \frac{1}{2} m \left(\dot{y} + \dot{\theta} L \sin\theta \right)^2 - mgy - mg(L - L\cos\theta) \right)$$
$$= m\left(\dot{\theta} L \cos\theta \right)\left(-\dot{\theta} L \sin\theta \right) + m\left(\dot{y} + \dot{\theta} L \sin\theta \right)(\dot{\theta} L \cos\theta) - mg(L\sin\theta)$$
$$= m\dot{y}\dot{\theta} L \cos\theta - mgL\sin\theta$$

Partial Derivative with respect to velocity, $\frac{\partial L}{\partial \dot{\theta}}$

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial}{\partial \dot{\theta}}\left(\frac{1}{2} m \left(\dot{\theta} L \cos\theta \right)^2 + \frac{1}{2} m \left(\dot{y} + \dot{\theta} L \sin\theta \right)^2 - mgy - mg(L - L\cos\theta) \right)$$
$$= mL^2\dot{\theta} + mL\dot{y}\sin\theta$$

Time Derivative of the Partial with respect to velocity, $\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}}$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{d}{dt}\left(mL^2\dot{\theta} + mL\dot{y}\sin\theta \right)$$
$$= mL^2\frac{d\dot{\theta}}{dt} + mL\sin\theta\frac{d\dot{y}}{dt} + mL\dot{y}\cos\theta\frac{d\theta}{dt}$$
$$= mL^2\ddot{\theta} + mL\ddot{y}\sin\theta + mL\dot{y}\dot{\theta}\cos\theta$$

Partial Derivative of the Raleigh Dissipation with respect to velocity,

$$\frac{\partial RD}{\partial \dot{\theta}} = 0$$

Combining these into the Lagrangian equation.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} + \frac{\partial RD}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = Q$$

$$\left\{ mL^2\ddot{\theta} + mL\ddot{y}\sin\theta + mL\dot{y}\dot{\theta}\cos\theta \right\} + \{0\} - \left\{ m\dot{y}\dot{\theta}L\cos\theta - mgL\sin\theta \right\} = 0$$

$$mL^2\ddot{\theta} + mL\ddot{y}\sin\theta + mL\dot{y}\dot{\theta}\cos\theta - m\dot{y}\dot{\theta}L\cos\theta + mgL\sin\theta = 0$$

$$\ddot{\theta} + \frac{1}{L}\ddot{y}\sin\theta + \frac{1}{L}\dot{y}\dot{\theta}\cos\theta - \frac{1}{L}\dot{y}\dot{\theta}\cos\theta + \frac{g}{L}\sin\theta = 0$$

$$\ddot{\theta} + \frac{1}{L}\ddot{y}\sin\theta + \frac{g}{L}\sin\theta = 0$$

For small angles: $\sin\theta = \theta$ the model equation becomes:

$$\ddot{\theta} + \left(\frac{\ddot{y}}{L} + \frac{g}{L} \right)\theta = 0$$