بسم الله الرحمن الرحيم

Mechanical Vibrations
ME445
Third Exam

Closed Book Exam
Time: 1 ½ hrs
Mon: 13/6/1424 H

Name:	Number:	Grade: ..50../50

1. For the string system shown in Figure (1), assume that its length is ℓ, mass per unit length is m and tension is T. One end of the string is fixed and the other end is connected to a mass M which moves in a frictionless slot. The mass M is also connected to a spring of stiffness k.

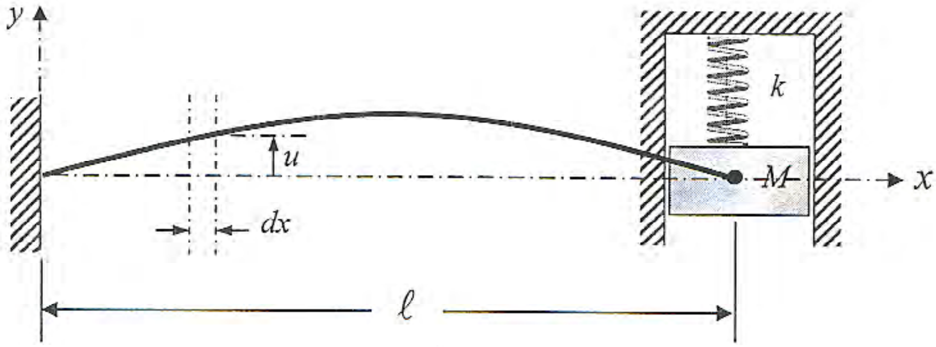
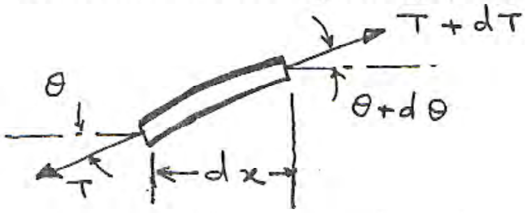


Figure (1)

a) Draw all the forces acting on an element of length dx. (2 points)



b) Write the equilibrium equation of the forces in the dynamic case. (2 points)

$$\sum F_y = \rho \, dx \, \frac{\partial^2 u(x,t)}{\partial t^2}$$

$$(T+dT)\sin(\theta+d\theta) - T\sin\theta = \rho \, dx \, \frac{\partial^2 u(x,t)}{\partial t^2}$$

OR: $(T+dT)(\theta+d\theta) - T\theta = \rho \, dx \, u_{tt}(x,t)$

c) Derive the equation of motion (wave equation) of the free transverse vibration of the string. Assume that the string is uniform and the tension is constant. (2 points)

$$\cancel{-T\theta} + \cancel{T\theta} + T\,d\theta + dT\,\theta + dT\,d\theta = \rho \, dx \, u_{tt}(x,t)$$

$$d[T\theta] + dT\,d\theta = \rho \, dx \, u_{tt}(x,t)$$

non linear part $\longrightarrow$ for linearization $\simeq 0$

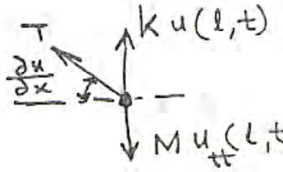
for constant T:

$T\,d\theta = \rho \, dx \, u_{tt}(x,t)$ But $d\theta = \frac{\partial \theta}{\partial x} dx = \frac{\partial^2 u}{\partial x^2} dx$

$$T \frac{\partial^2 u(x,t)}{\partial x^2} \cancel{dx} = \rho \cancel{dx} \, u_{tt}(x,t) \Rightarrow c^2 u_{xx}(x,t) = u_{tt}(x,t)$$

d) Derive the associated boundary conditions. (2 points)

at $x = 0$ $\qquad u(0,t) = 0$



at $x = l$ $\qquad T \dfrac{\partial u(l,t)}{\partial x} + k\, u(l,t) = M \dfrac{\partial^2 u(l,t)}{\partial t^2}$

e) Solve EXACTLY the wave equation using the method of separation of variables.
(5 points)

$$u(x,t) = \left[A \cos\left(\frac{\omega x}{c}\right) + B \sin\left(\frac{\omega x}{c}\right)\right]\left[C \cos(\omega t) + D \sin(\omega t)\right]$$

1^{st} B.C. , $u(0,t) = 0 \implies \boxed{A = 0}$

$$\left.\begin{aligned} \therefore u(x,t) &= B \sin\left(\frac{\omega x}{c}\right)\left[C \cos(\omega t) + D \sin(\omega t)\right] \\ \frac{\partial u(x,t)}{\partial x} &= B\left(\frac{\omega}{c}\right) \cos\left(\frac{\omega x}{c}\right)\left[C \cos(\omega t) + D \sin(\omega t)\right] \\ \frac{\partial^2 u(x,t)}{\partial t^2} &= -\omega^2 B \sin\left(\frac{\omega x}{c}\right)\left[C \cos(\omega t) + D \sin(\omega t)\right] \end{aligned}\right\} \rightarrow (*)$$

subs eq$^{\underline{n}}$ (*) into the second B.C. when $x = l$,

$$T \cancel{B}\left(\frac{\omega}{c}\right) \cos\left(\frac{\omega l}{c}\right) \cancel{\phi(t)} + k \cancel{B} \sin\left(\frac{\omega l}{c}\right) \cancel{\phi(t)} = -M \cancel{B} \omega^2 \sin\left(\frac{\omega l}{c}\right) \cancel{\phi(t)}$$

$$\frac{T\omega}{c} \cos\left(\frac{\omega l}{c}\right) + \left(k + M\omega^2\right) \sin\left(\frac{\omega l}{c}\right) = 0$$

f) Derive the characteristics equation of the system. (4 points)

As, $\frac{T\omega}{c}\cos\left(\frac{\omega\ell}{c}\right) + (k + M\omega^2)\sin\left(\frac{\omega\ell}{c}\right) = 0$

$$\Rightarrow \boxed{\tan\left(\frac{\omega\ell}{c}\right) + \frac{T\omega}{(k+M\omega^2)c} = 0}$$

The characteristics eq$^{\underline{n}}$ of the string

where : $c = \sqrt{\frac{T}{\rho}}$ or $c = \sqrt{\frac{T}{m}}$

MODE SHAPE EQ$\underline{N}$ is: $Y(x) = \underbrace{\cancel{B}}_{=1} \sin\left(\frac{\omega x}{c}\right)$

g) Extract the first three natural frequencies of the string and draw their mode shapes. Assume m=1, M=1, k=1, T=1, and ℓ =1. (3 points)

$c = 1$

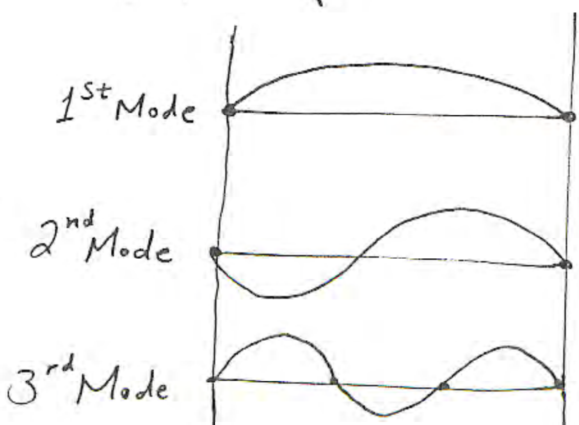
$$\therefore \tan\omega + \frac{\omega}{1+\omega^2} = 0$$

this equation can be solved graphacilly to extract ω_n's.

$\omega_1 = 1.6$

$\omega_2 = 4.7$

$\omega_3 = 7.8$



2. Consider a rod in axial vibration attached to a spring of stiffness k at $x=0$ and free at $x=\ell$ as shown in Figure (2). The rod is subjected to the force per unit length $f(x,t)$, its mass per unit length is $m(x)$ and its axial stiffness is $EA(x)$, where E is the modulus of elasticity and $A(x)$ the cross-sectional area.

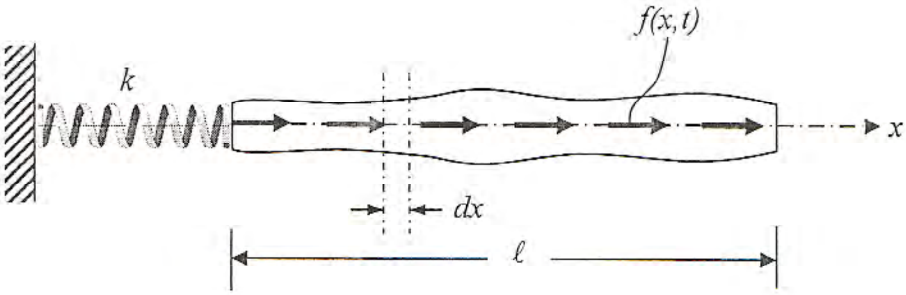
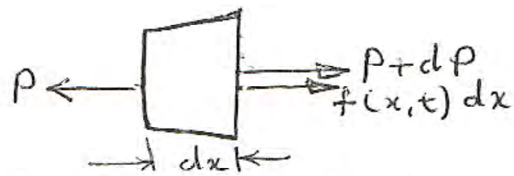


Figure (2)

a) Draw all the forces acting on an element of length dx. (2 points)



b) Write the equilibrium equation of the forces in the dynamic case. (2 points)

$$(P+dP) + f\,dx - P = \rho A\,dx\,\frac{\partial^2 u}{\partial t^2}$$

c) Derive the equation of motion for the rod in the axial vibration. (2 points)

$$P = \sigma A = EA\frac{\partial u}{\partial x}$$

& $dP = \frac{\partial P}{\partial x}dx$

$$\cancel{EA\frac{\partial u}{\partial x}} + \frac{\partial}{\partial x}\left[EA\frac{\partial u}{\partial x}\right]\cancel{dx} + f\,\cancel{dx} - \cancel{EA\frac{\partial u}{\partial x}} = \rho A\,\cancel{dx}\,\frac{\partial^2 u}{\partial t^2}$$

$$\therefore\ \frac{\partial}{\partial x}\left[EA(x)\,\frac{\partial u(x,t)}{\partial x}\right] + f(x,t) = \rho(x)A(x)\,\frac{\partial^2 u}{\partial t^2}$$

d) Derive the associated boundary conditions. (2 points)

at $x=0$ $\quad AE\dfrac{\partial u(0,t)}{\partial x} = k\,u(0,t)$

at $x=l$ $\quad \dfrac{\partial u(l,t)}{\partial x} = 0$

e) If the rod is uniform and under free vibration, what's the characteristics equation? (4 points)

$$E\cancel{A}\frac{\partial^2 u(x,t)}{\partial x^2} = \rho\cancel{A}\frac{\partial^2 u(x,t)}{\partial t^2}$$

$$c^2 u_{xx} = u_{tt} \qquad \text{where } c^2 = \frac{E}{\rho}$$

$$u(x,t) = \left[C_1\cos\left(\frac{\omega x}{c}\right) + C_2\sin\left(\frac{\omega x}{c}\right)\right]\phi(t)$$

$$\frac{\partial u(x,t)}{\partial x} = \left[-\left(\frac{\omega}{c}\right)C_1\sin\left(\frac{\omega x}{c}\right) + \left(\frac{\omega}{c}\right)C_2\cos\left(\frac{\omega x}{c}\right)\right]\phi\, t$$

$$u(0,t) = C_1\,\phi(t) \quad \& \quad \frac{\partial u(0,t)}{\partial x} = \frac{\omega}{c}C_2\,\phi(t)$$

$$\therefore \frac{AE\omega}{c}C_2 = kC_1 \Rightarrow \boxed{C_1 = \frac{AE\omega}{ck}C_2}$$

$\therefore$ Characteristics Eqn is :

$$-\frac{AE\omega}{ck}\sin\left(\frac{\omega l}{c}\right) + \cos\left(\frac{\omega l}{c}\right) = 0 \Rightarrow \boxed{\tan\left(\frac{\omega l}{c}\right) = \frac{ck}{AE\omega}} \longrightarrow (*)$$

f) Plot the three lowest modes of vibration. Assume m=1, E=1, A=1, k=1, and ℓ=1. (3 points)

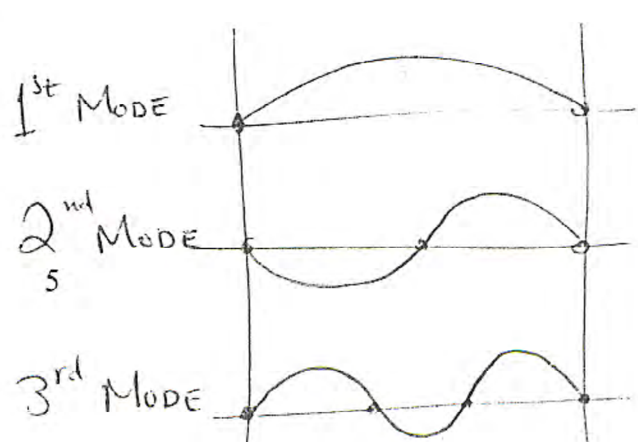
Eqn (*) can be solved numerically or we can solve it graphically

$$\tan(\omega) - \frac{1}{\omega} = 0$$

$\omega_1 \simeq 1.6$ Hz

$\omega_2 \simeq 4.7$ Hz

$\omega_3 \simeq 7.8$ Hz



3. A uniform beam of length ℓ, mass per unit length m, Young's modulus E, area of cross section A, and area of moment of inertia I is subjected to transverse vibration. The beam is fixed at x=0 and the other end is connected to a mass M. The mass is also connected to a damper of damping constant c as shown in Figure (3).

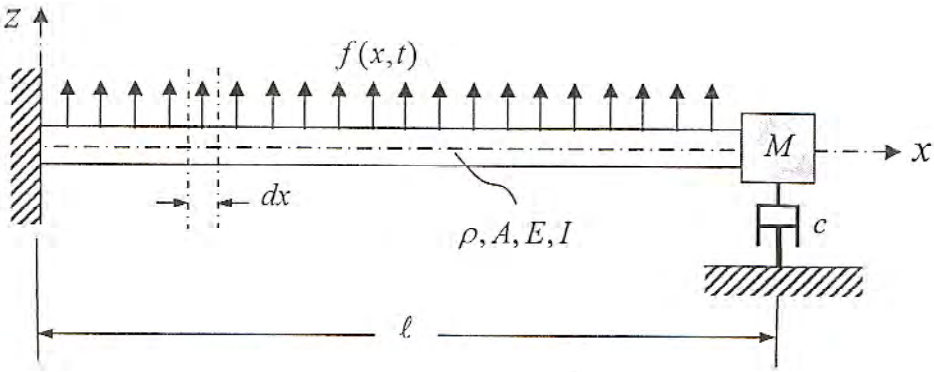
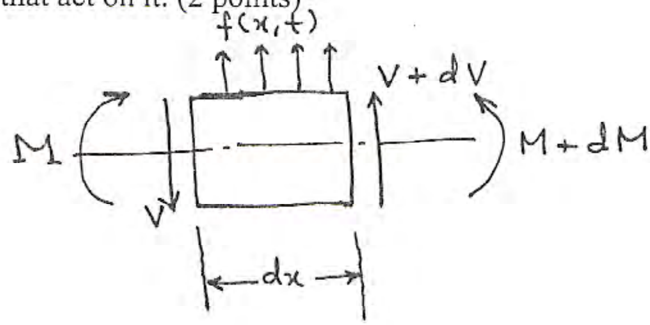


Figure (3)

a) Draw a free body diagram corresponding to the beam differential element of length dx and show all the loads that act on it. (2 points)



b) Write the force equation of motion in the z direction. (2 points)

$$\Sigma F = \rho A(x)\, dx \,\frac{\partial^2 w(x,t)}{\partial t^2}$$

$$= -\cancel{V} + f(x,t)\, dx + (\cancel{V} + dV)$$

$$= dV + f(x,t)\, dx = \rho A(x)\, dx \,\frac{\partial^2 w(x,t)}{\partial t^2}$$

c) Show that $V = \frac{\partial M}{\partial x}$ using the moment equation of motion about y axis. (4 points)

$$(\cancel{M} + dM) - (V + dV)dx + f(x,t)\, dx \frac{dx}{2} - \cancel{M} = 0$$

$$dM - V\,dx - \underbrace{dV\,dx}_{\text{non linear} \simeq 0} + \underbrace{f(x,t)\,dx\,\frac{dx}{2}}_{\text{non linear} = 0} = 0$$

$$\frac{\partial M}{\partial x} \cancel{dx} - V \cancel{dx} = 0$$

$$\Longrightarrow \quad V = \frac{\partial M}{\partial x}$$

d) Derive the partial differential equation for the bending vibration of the beam. (3 points)

from b) : $\quad dV + f(x,t)\,dx = \rho A\,dx \frac{\partial^2 w(x,t)}{\partial t^2}$

But $dV = \frac{\partial V}{\partial x} dx$ & from c) $V = \frac{\partial M}{\partial x}$

$$\therefore \frac{\partial}{\partial} \left[\frac{\partial M}{\partial x}\right] + f(x,t) = \rho A \frac{\partial^2 w(x,t)}{\partial t^2}$$

But $\quad M = EI \frac{\partial^2 w(x,t)}{\partial x^2}$

$$\therefore \frac{\partial^2}{\partial x^2}\left[EI \frac{\partial^2 w(x,t)}{\partial x^2}\right] + f(x,t) = \rho A \frac{\partial^2 w(x,t)}{\partial t^2}$$

For uniform beam & free vibration: $c^2 w_{xxxx} = w_{tt}$ $\quad c^2 = \frac{EI}{\rho A}$

e) Express the boundary conditions of the beam? (4 points)

at $x = 0$, $\quad w(0,t) = 0$ & $w_x(0,t) = 0$

at $x = L$, $\quad EI\, w_{xxx}(l,t) = C\, w_t + M\, w_{tt}$

& $\quad EI\, w_{xx}(l,t) = 0$

مع دعواتي لكم بالتوفيـــق

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