MENG366

Mathematical Modeling of Mechanical and Electrical Systems

Prof. Saeed Asiri
saeed@asiri.net

Translational Mechanical Systems

- Basic (Idealized) Modeling Elements
- Interconnection Relationships -Physical Laws
- Derive Equation of Motion (EOM) - SDOF
- Energy Transfer
- Series and Parallel Connections
- Derive Equation of Motion (EOM) - MDOF

Key Concepts to Remember

- Three primary elements of interest
  - Mass (inertia) *M*
  - Stiffness (spring) *K*
  - Dissipation (damper) *B*
  - Usually we deal with “equivalent” *M, K, B*
    - *Distributed mass → lumped mass*
- Lumped parameters
  - Mass maintains motion (Kinetic Energy)
  - Stiffness restores motion (Potential Energy)
  - Damping eliminates motion (~~Eliminate Energy~~ ?) (Damp Energy)

Variables

- ***x*** : **displacement** [m]
- ***v*** : **velocity** [m/sec]
- ***a*** : **acceleration** [m/sec^2]
- ***f*** : **force** [N]
- ***p*** : **power** [Nm/sec]
- ***w*** : **work** (energy) [Nm]

1 [Nm] = 1 [J] (Joule)

$$v = \dot{x} = \frac{d}{dt}x$$

$$a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{d}{dt}x\right) = \frac{d^2}{dt^2}x = \ddot{x}$$

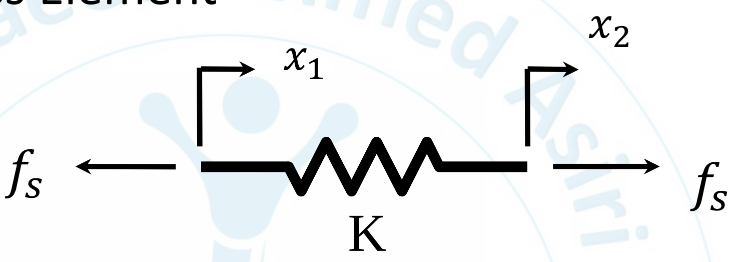
$$p = f \cdot v = f \cdot \dot{x} = \frac{d}{dt}w$$

$$w(t_1) = w(t_0) + \int_{t_0}^{t_1} p(t)\ dt$$

$$= w(t_0) + \int_{t_0}^{t_1} (f \cdot \dot{x})\ dt$$

Basic (Idealized) Modeling Elements

- **Spring**
  - Stiffness Element



$$f_s = K\,\underbrace{(x_2 - x_1)}_{\delta x}$$

  - Idealization
    - Massless
    - No Damping
    - Linear
  - Stores Energy

    Potential Energy

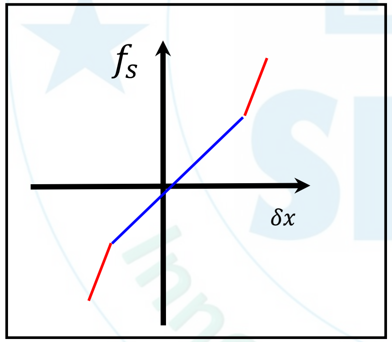
$$U = \frac{1}{2}k(\delta x)^2$$

- Reality
  - 1/3 of the spring mass may be considered into the lumped model.
  - In large displacement operation springs are *nonlinear*.

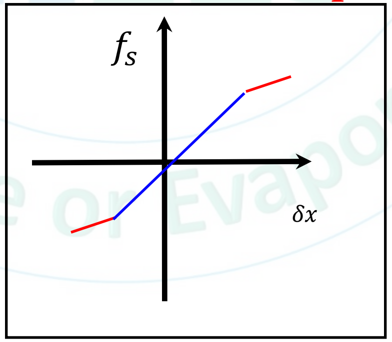
Linear spring

→ *nonlinear spring*

→ *broken spring !!*



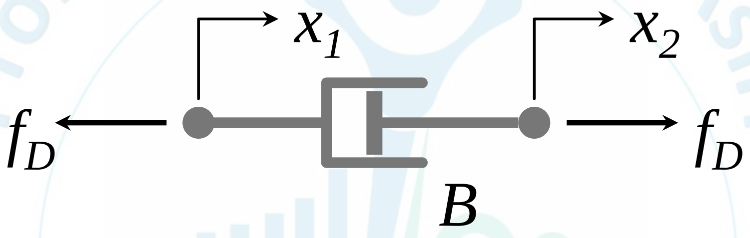
Hard Spring



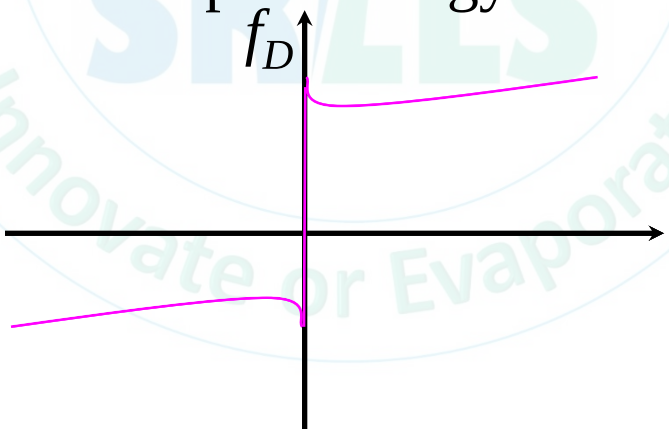
Soft Spring

Basic Modeling Elements

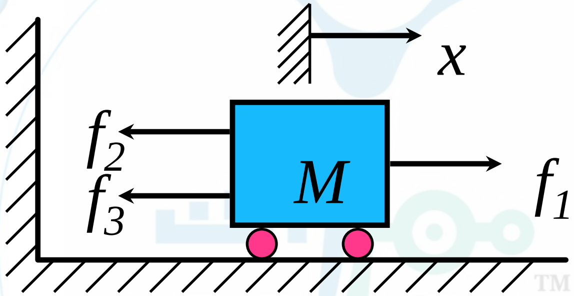
- **Damper**
  - Friction Element



  - Dissipate Energy



- **Mass**
  - Inertia Element



$$M\ddot{x} = \sum_i f_i = f_1 - f_2 - f_3$$

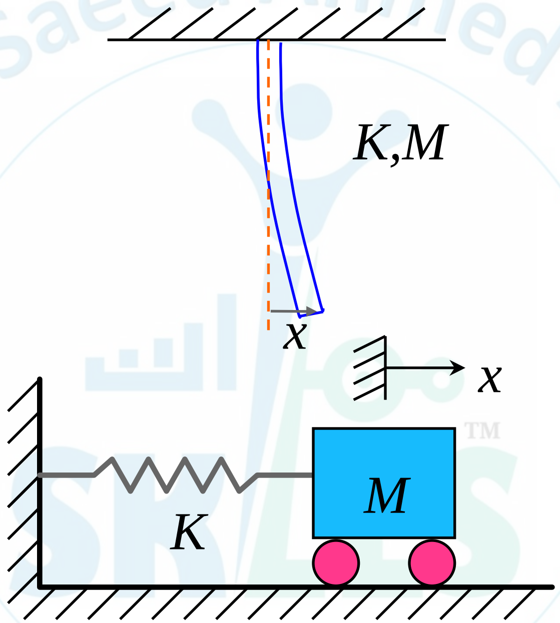
  - Stores Kinetic Energy

$$T = \frac{1}{2}M\dot{x}^2$$

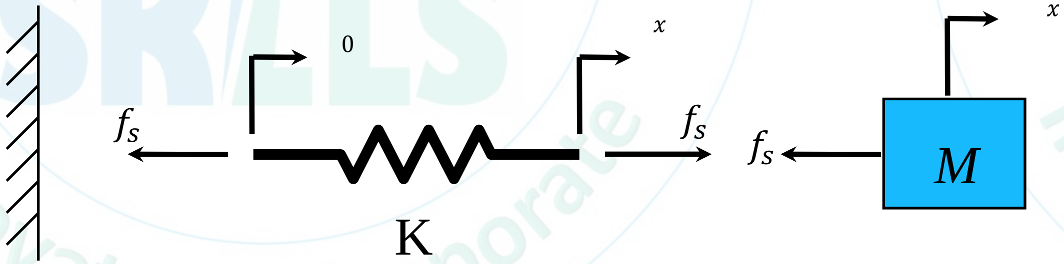
Interconnection Laws

- Newton's Second Law

Lumped Model of a Flexible Beam



- Newton's Third Law
  - Action & Reaction Forces



$f_s = K(x-0) = Kx$

Massless spring

$M\ddot{x} = -f_s$

$M\ddot{x} = -Kx$

$$\boxed{M\ddot{x} + Kx = 0}$$

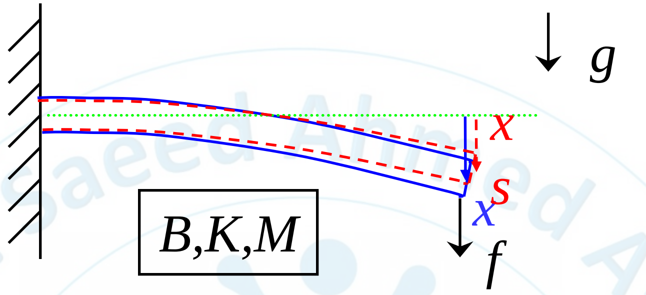
E.O.M.

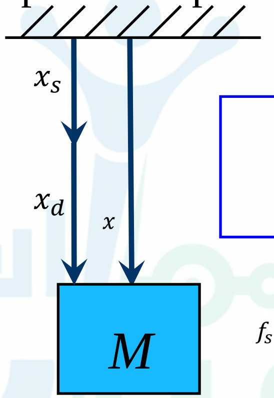
Modeling Steps

- **Understand System Function, Define Problem, and Identify Input/Output Variables**
- **Draw Simplified Schematics Using Basic Elements**
- **Develop Mathematical Model (Diff. Eq.)**
  - Identify reference point and positive direction.
  - Draw Free-Body-Diagram (FBD) for each basic element.
  - Write Elemental Equations as well as Interconnecting Equations by applying physical laws. (*Check: # eq = # unk)*
  - Combine Equations by eliminating intermediate variables.
- **Validate Model by Comparing Simulation Results with Physical Measurements**

Single Degree of Freedom (SDOF) System

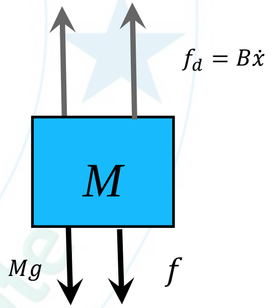
- **Define Problem** **The motion of the object**
- **Input** f
- **Output** x
- **Develop Mathematical Model (Diff. Eq.)**
  - Identify reference point and positive direction.

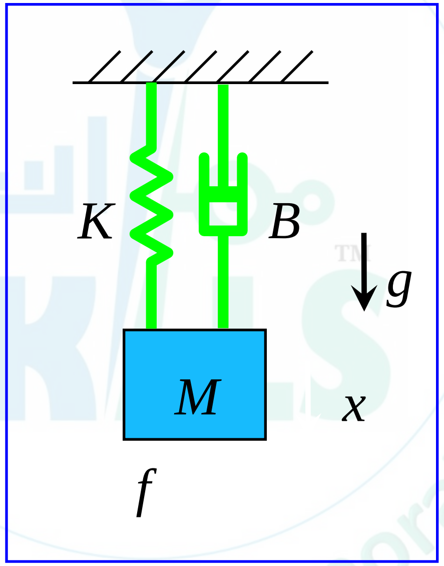




$x = x_s + x_d$

$Kx_s = Mg$





  - Draw Free-Body-Diagram (FBD)
  - Write Elemental Equations

$M\ddot{x} = -B\dot{x} - Kx + Mg + f$

$M\ddot{x} + B\dot{x} + Kx = Mg + f$

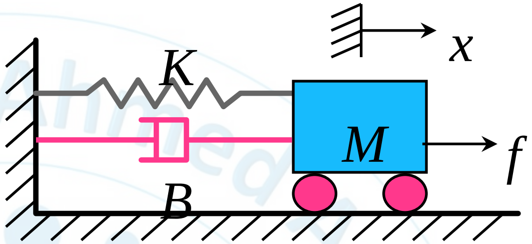
$M\ddot{x}_u + B\dot{x}_u + Kx_u = Mg + f$ *From the undeformed position*

$M\ddot{x}_d + B\dot{x}_d + Kx_d = f$ *From the deformed (static equilibrium) position*

- **Validate Model by Comparing Simulation Results with Physical Measurement**

Energy Distribution

- **EOM of a simple Mass-Spring-Damper System**



We want to look at the energy distribution of the system. How should we start ?

- **Multiply the above equation by the velocity term *v* :** $\Leftarrow$ ***What have we done ?***

$$M\ddot{x}\cdot\dot{x} + B\dot{x}\cdot\dot{x} + Kx\cdot\dot{x} = f(t)\cdot\dot{x}$$

- **Integrate the second equation w.r.t. time:** $\Leftarrow$ ***What are we doing now ?***

$$\underbrace{\int_{t_0}^{t_1} M\ddot{x}\cdot\dot{x}\ dt}_{\Delta KE} + \underbrace{\int_{t_0}^{t_1} B\dot{x}\cdot\dot{x}\ dt}_{\int_{t_0}^{t_1} B\dot{x}^2 dt\ \geq\ 0} + \underbrace{\int_{t_0}^{t_1} Kx\cdot\dot{x}\ dt}_{\Delta PE} = \underbrace{\int_{t_0}^{t_1} f(t)\cdot v\ dt}_{W}$$

$\Downarrow$ Change of kinetic energy

$\Downarrow$ Energy dissipated by damper

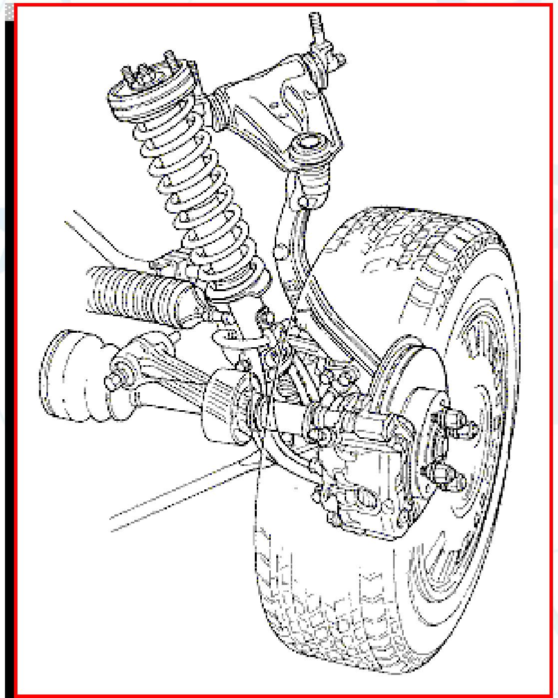
$\Downarrow$ Change of potential energy

Total work done by the applied force $f(t)$ from time t_0 to t_1

SDOF Suspension (Example)

- **Suspension System**

  Minimize the effect of the surface roughness of the road on the driver's comfort.



Double Wishbone Suspension

– **Simplified Schematic (neglecting tire model)**

From the "absolute zero"

$$\boxed{M\ddot{x} + B(\dot{x} - \dot{x}_p) + K(x - x_p) = -Mg}$$

From the path

$$\boxed{M\ddot{x}_{op} + B\dot{x}_{op} + Kx_{op} = -Mg - M\ddot{x}_p}$$

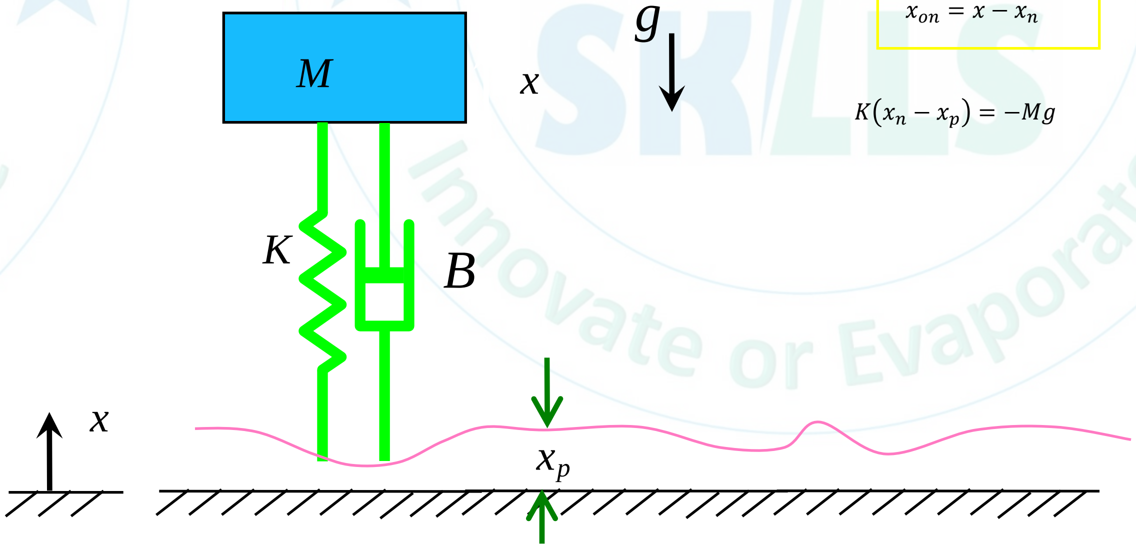
$$\boxed{x_{op} = x - x_p}$$

From nominal position

$$\boxed{M\ddot{x}_{on} + B\dot{x}_{on} + Kx_{on} = -M\ddot{x}_p}$$

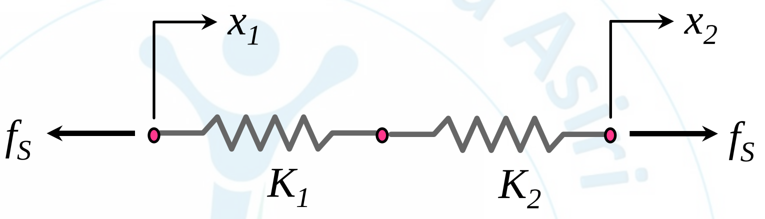
$$\boxed{x_{on} = x - x_n}$$

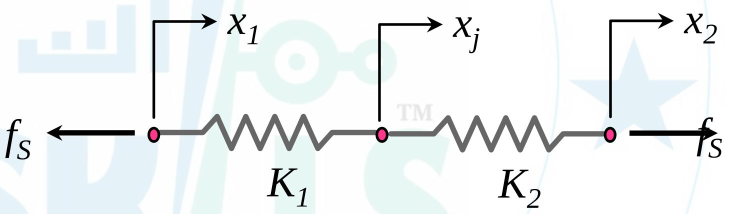
$$K(x_n - x_p) = -Mg$$



Series Connection

- Springs in Series





$$K_1\left(x_j - x_1\right) = K_2\left(x_2 - x_j\right)$$

$$x_j = \frac{1}{K_1 + K_2}\left[K_2 x_2 + K_1 x_1\right]$$

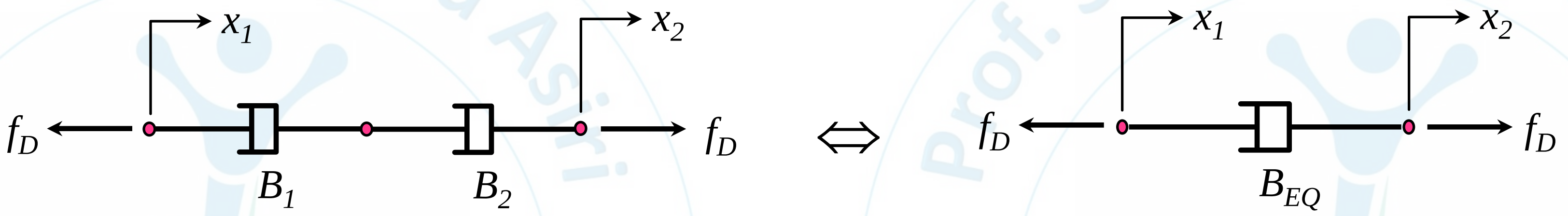
$$f_s = K_1\left(x_j - x_1\right)$$

$$= K_1\left\{\underbrace{\frac{1}{K_1 + K_2}\left[K_2 x_2 + K_1 x_1\right]}_{x_j} - x_1\right\}$$

$$f_s = \boxed{\underbrace{\frac{K_1 K_2}{K_1 + K_2}}_{K_{eq}}}\left[x_2 - x_1\right]$$

Series Connection

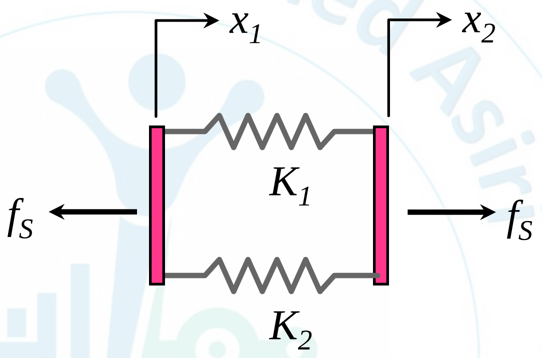
- Dampers in Series



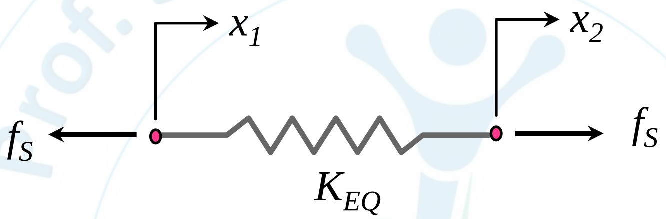
$$f_d = \underbrace{\frac{B_1 B_2}{B_1 + B_2}}_{B_{eq}} [\dot{x}_2 - \dot{x}_1]$$

Parallel Connection

- Springs in Parallel



$\Leftrightarrow$

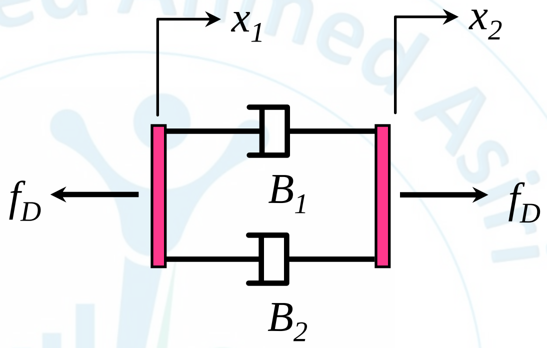


$$f_s = K_1(x_2 - x_1) + K_2(x_2 - x_1)$$

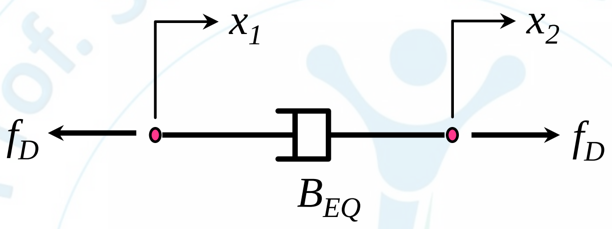
$$\boxed{f_s = \underbrace{(K_1 + K_2)}_{K_{eq}}(x_2 - x_1)}$$

Parallel Connection

Dampers in Parallel

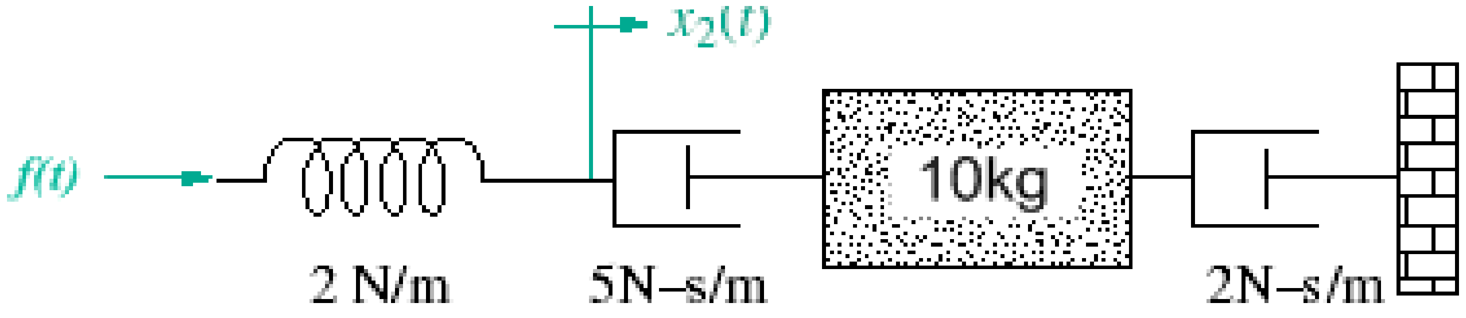


$\Leftrightarrow$

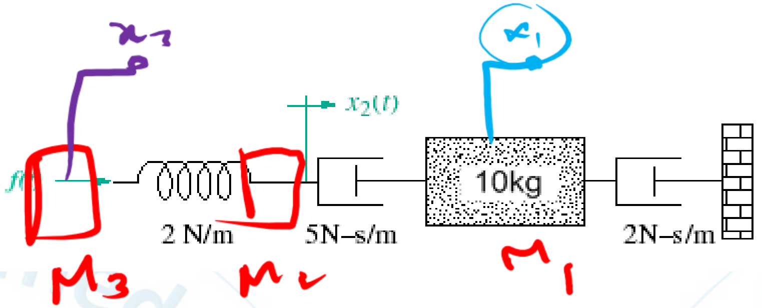


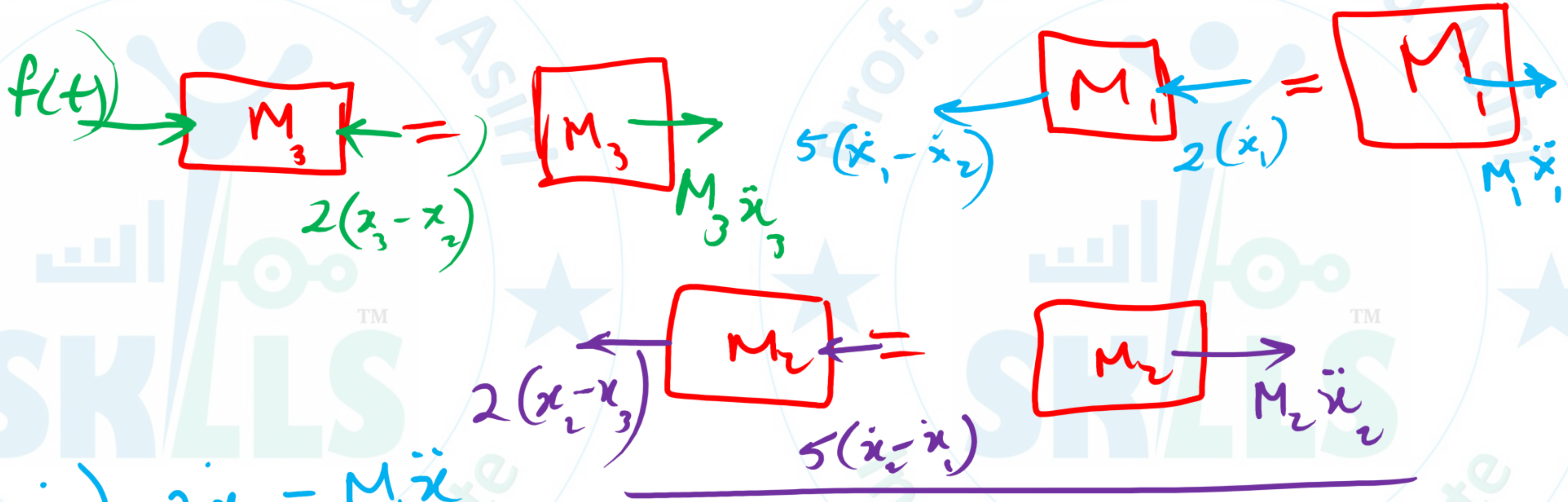
$$f_s = \boxed{\underbrace{(B_1 + B_2)}_{K_{eq}}}\,(\dot{x}_2 - \dot{x}_1)$$

Example 1



Example 1





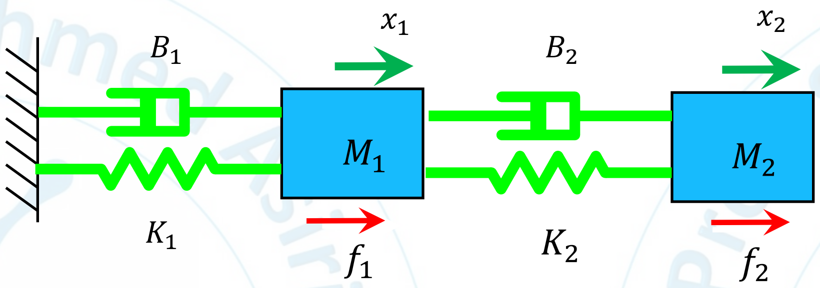
$-5(\dot{x}_1-\dot{x}_2)-2\dot{x}_1 = M_1\ddot{x}_1$

$-2(x_2-x_3)-5(\dot{x}_2-\dot{x}_3) = M_2\ddot{x}_2 \rightarrow 0$

$f(t)-2(x_3-x_2) = M_3\ddot{x}_3 \rightarrow 0$

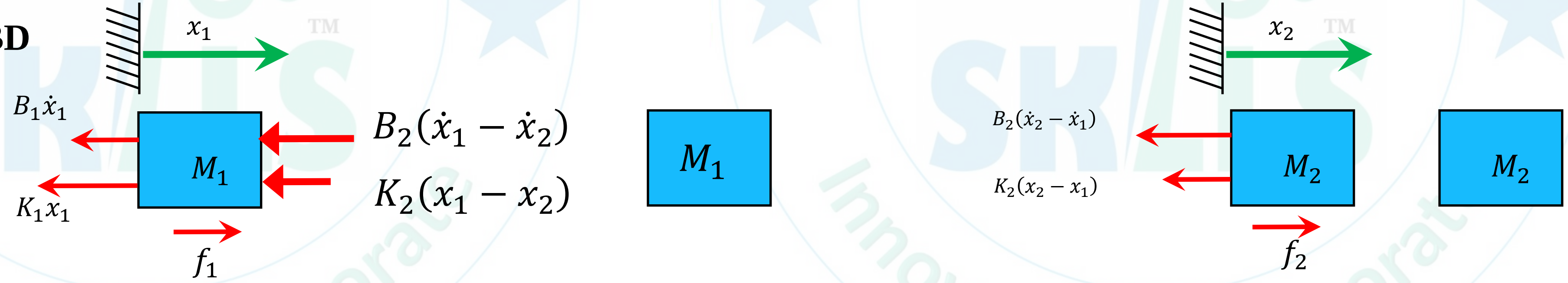
Two Degree of Freedom (TDOF) System

- DOF = 2



- Absolute coordinates

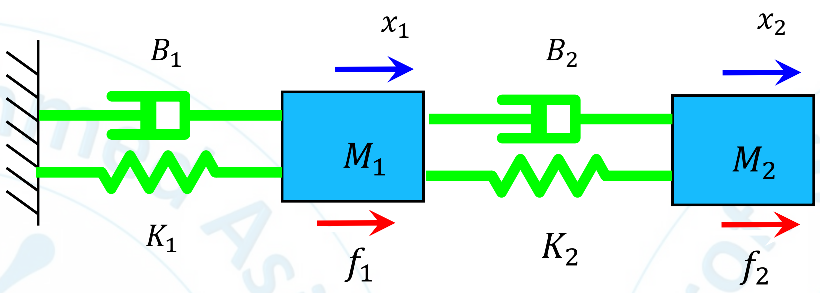
- FBD



- Newton's law

$$M_1\ddot{x}_1 = -B_1\dot{x}_1 - K_1x_1 + B_2(\dot{x}_2 - \dot{x}_1) + K_2(x_2 - x_1) + f_1(t)$$
$$M_2\ddot{x}_2 = -B_2(\dot{x}_2 - \dot{x}_1) - K_2(x_2 - x_1) + f_2(t)$$

Two Degree of Freedom (TDOF) System



Static coupling

- **Absolute coordinates**

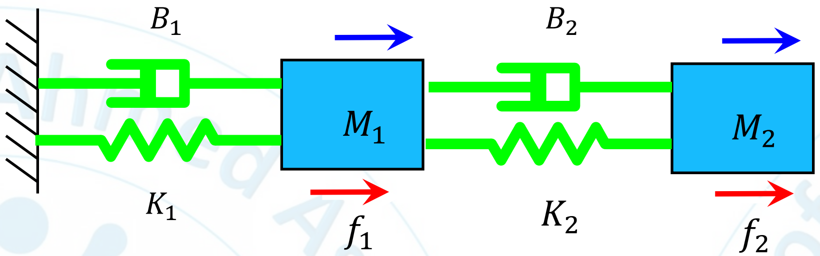
$$M_1\ddot{x}_1 + (B_1 + B_2)\dot{x}_1 + (K_1 + K_2)x_1 - B_2\dot{x}_2 - K_2x_2 = f_1(t)$$
$$M_2\ddot{x}_2 + B_2\dot{x}_2 + K_2x_2 - B_2\dot{x}_1 - K_2x_1 = f_2(t)$$

- **Relative coordinates**

$$M_1\ddot{x}_1 + B_1\dot{x}_1 + K_1x_1 - B_2\dot{x}_{21} - K_2x_{21} = f_1(t)$$
$$M_2\ddot{x}_1 + M_2\ddot{x}_{21} + B_2\dot{x}_{21} + K_2x_{21} = f_2(t)$$

Dynamic coupling

Two DOF System – Matrix Form of EOM



- **Absolute coordinates**

SYMMETRIC

Mass matrix | Damping matrix | Stiffness matrix | Output vector | Input vector

$$\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} B_1 + B_2 & -B_2 \\ -B_2 & B_2 \end{bmatrix}\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} K_1 + K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} f_1(t) \\ f_2(t) \end{Bmatrix}$$

- **Relative coordinates**

NON-SYMMETRIC

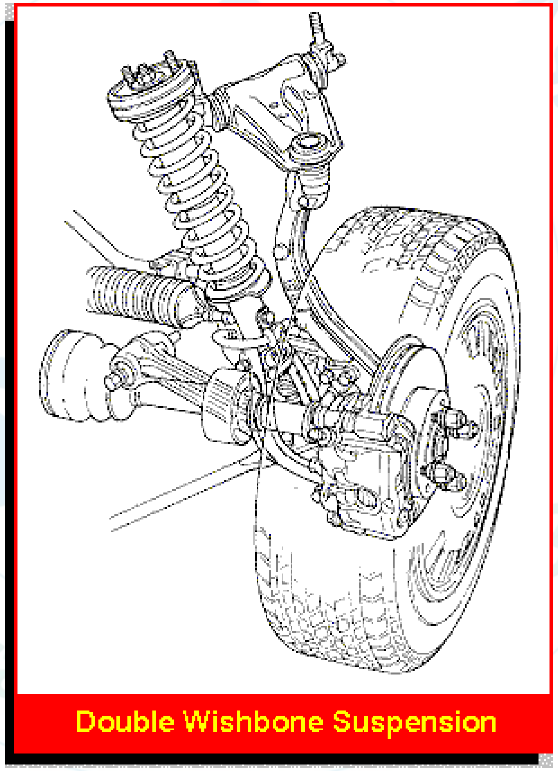
$$\begin{bmatrix} M_1 & 0 \\ M_2 & M_2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_{21} \end{Bmatrix} + \begin{bmatrix} B_1 & -B_2 \\ 0 & B_2 \end{bmatrix}\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_{21} \end{Bmatrix} + \begin{bmatrix} K_1 & -K_2 \\ 0 & K_2 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_{21} \end{Bmatrix} = \begin{Bmatrix} f_1(t) \\ f_2(t) \end{Bmatrix}$$

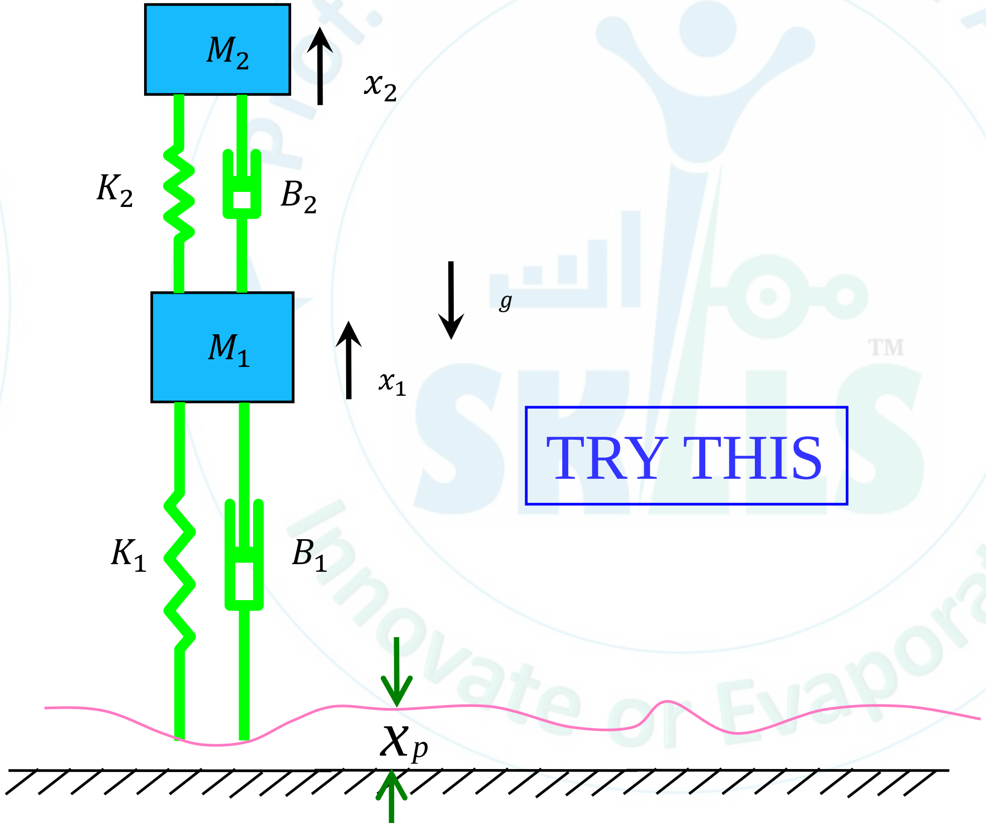
MDOF Suspension

Example 2

- **Suspension System**

  – Simplified Schematic (with tire model)

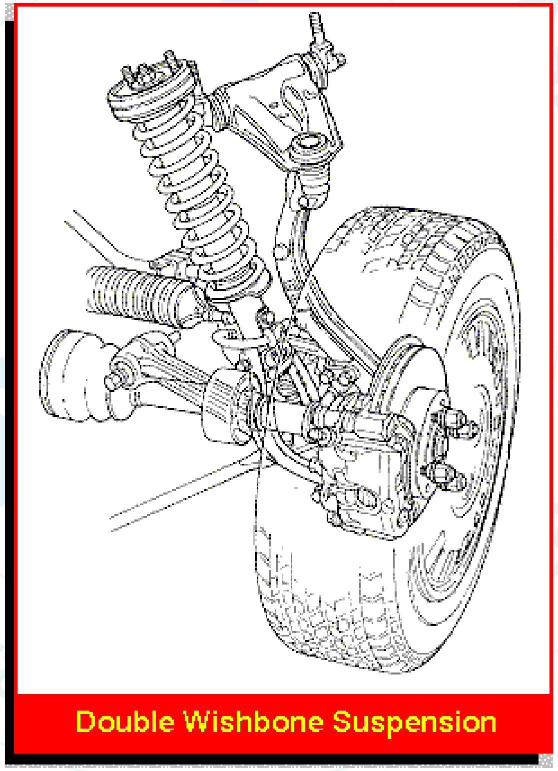




TRY THIS

MDOF Suspension

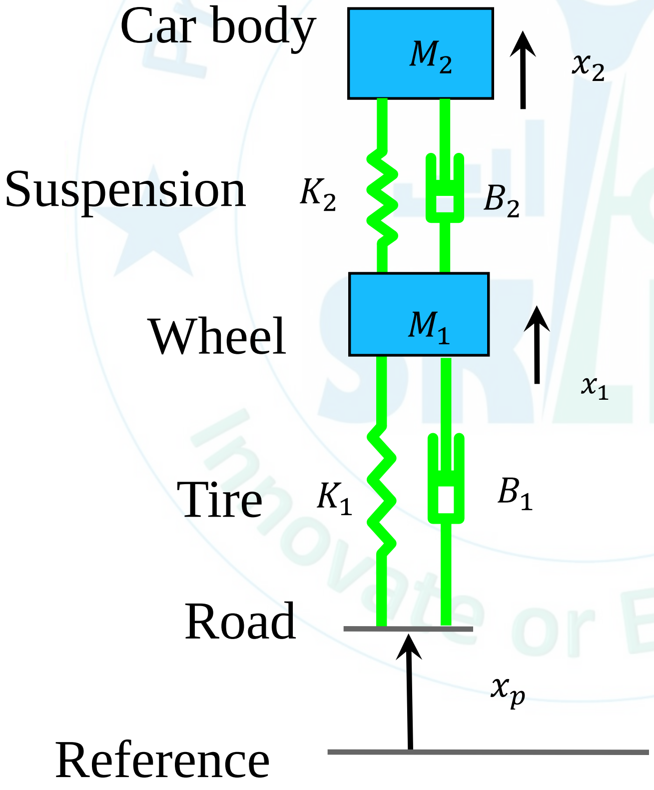
- **Suspension System**



Double Wishbone Suspension

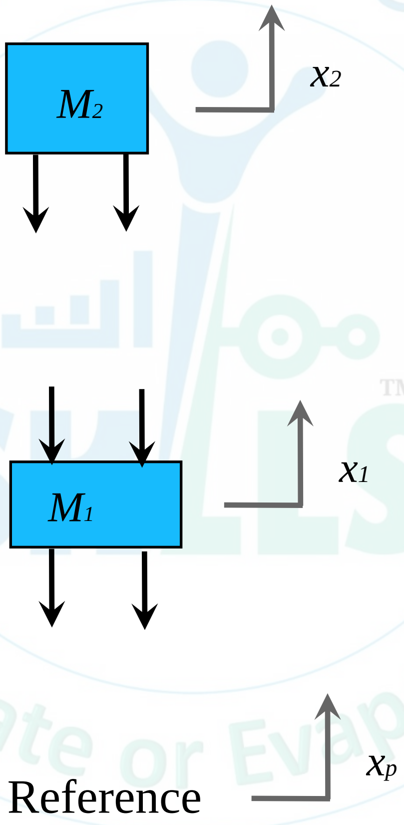
– Simplified Schematic (with tire model)

Assume ref. is when springs are Deflected by weights



MDOF Suspension

– **Draw FBD**



– **Apply Newton's 2nd Laws**

$$M_2\ddot{x}_2 + B_2\dot{x}_2 - B_2\dot{x}_1 + K_2x_2 - K_2x_1 = 0$$
$$\Rightarrow M_1\ddot{x}_1 - B_2\dot{x}_2 + (B_2 + B_1)\dot{x}_1 - K_2x_2 + (K_2 + K_1)x_1 = B_1\dot{x}_p + K_1x_p$$

MDOF Suspension

- **Matrix Form**

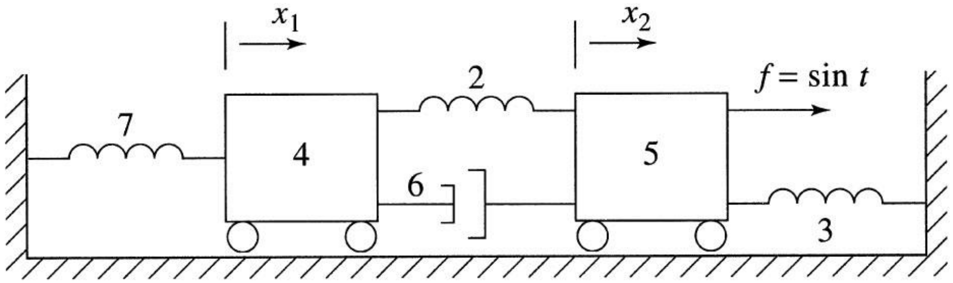
$$M_2\ddot{x}_2 + B_2\dot{x}_2 - B_2\dot{x}_1 + K_2x_2 - K_2x_1 = 0$$
$$M_1\ddot{x}_1 - B_2\dot{x}_2 + (B_2 + B_1)\dot{x}_1 - K_2x_2 + (K_2 + K_1)x_1 = B_1\dot{x}_p + K_1x_p$$

Define vector $x = \begin{Bmatrix} x_2 \\ x_1 \end{Bmatrix}$

$$\underbrace{\begin{bmatrix} M_2 & 0 \\ 0 & M_1 \end{bmatrix}}_{\text{Mass matrix}}\begin{Bmatrix} \ddot{x}_2 \\ \ddot{x}_1 \end{Bmatrix} + \underbrace{\begin{bmatrix} B_2 & -B_2 \\ -B_2 & B_2 + B_1 \end{bmatrix}}_{\text{Damping matrix}}\begin{Bmatrix} \dot{x}_2 \\ \dot{x}_1 \end{Bmatrix} + \underbrace{\begin{bmatrix} K_2 & -K_2 \\ -K_2 & K_2 + K_1 \end{bmatrix}}_{\text{Stiffness matrix}}\begin{Bmatrix} x_2 \\ x_1 \end{Bmatrix} = \underbrace{\begin{Bmatrix} 0 \\ B_1\dot{x}_p + K_1x_p \end{Bmatrix}}_{\text{Input Vector}}$$

Two Degree of Freedom (TDOF) System

Example 3



(a)

(b)

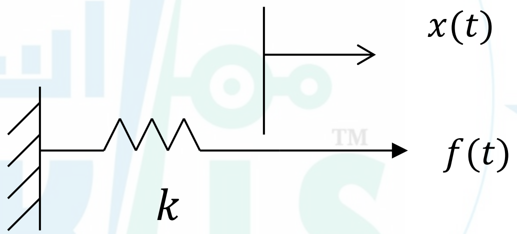
$$7x_1 + 4\frac{d^2x_1(t)}{dt^2} + 2(x_1 - x_2) + 6\frac{d(x_1 - x_2)}{dt} = 0$$

$$2(x_2 - x_1) + 6\frac{d(x_2 - x_1)}{dt} + 5\frac{d^2x_2}{dt^2} + 3x_2 = \sin t$$

Rotational Mechanical Systems

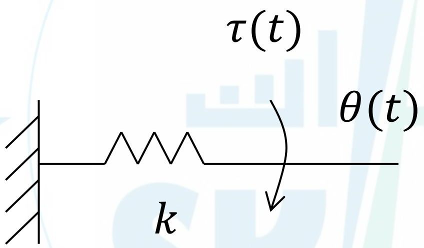
Translational mechanical components

spring



$$f(t) = kx(t)$$

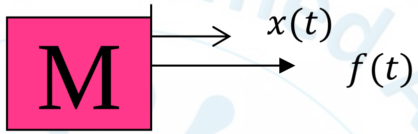
Rotational mechanical components



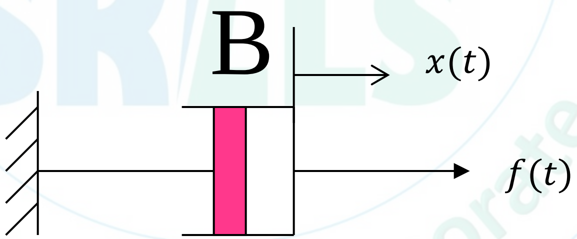
$$\tau(t) = k\theta(t)$$

Rotational Mechanical Systems

Translational mechanical components

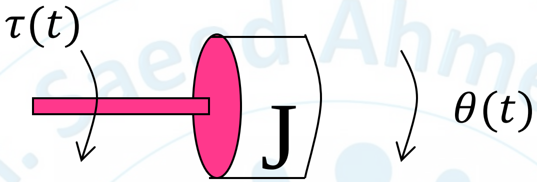


$$f(t) = Ma = Mx''(t)$$

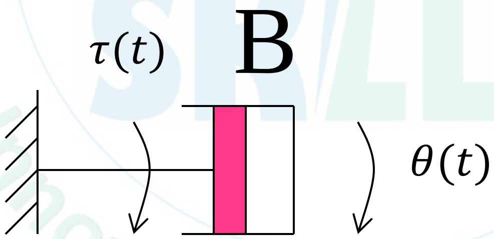


$$f(t) = Bv(t) = Bx'(t)$$

Rotational mechanical components



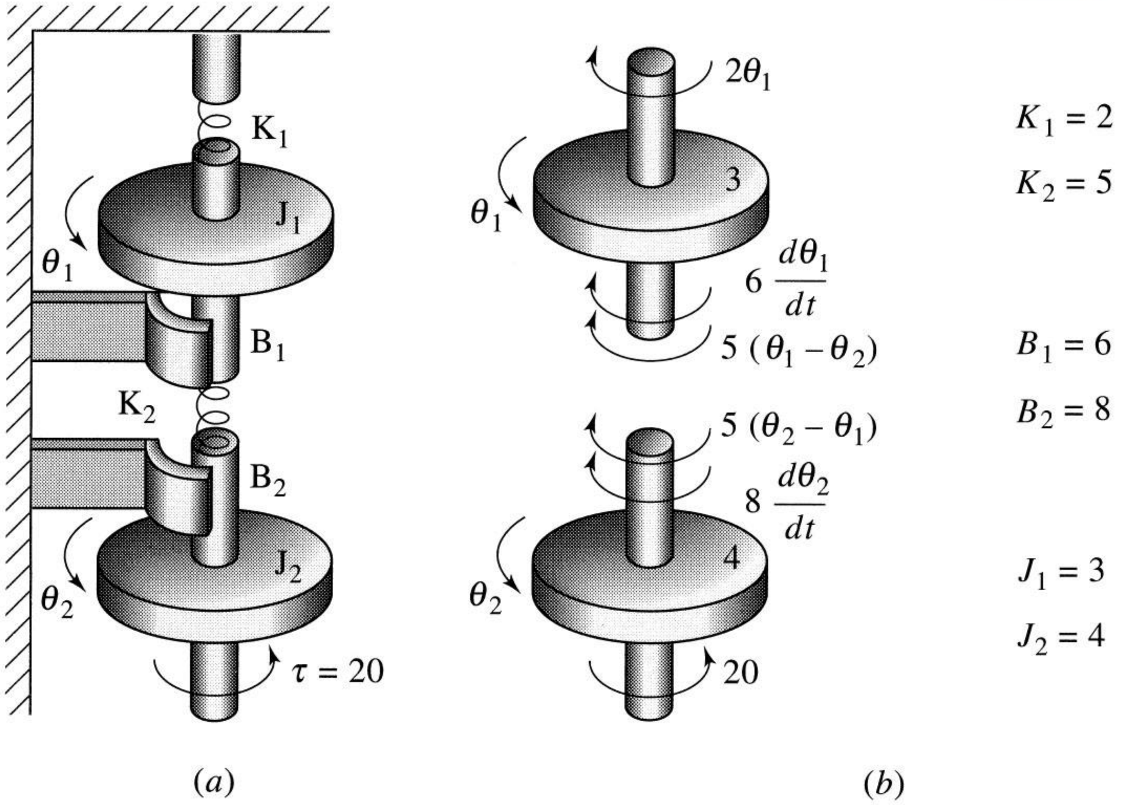
$$\tau(t) = J\theta''(t)$$



$$\tau(t) = B\theta'(t)$$

Rotational Mechanical Systems

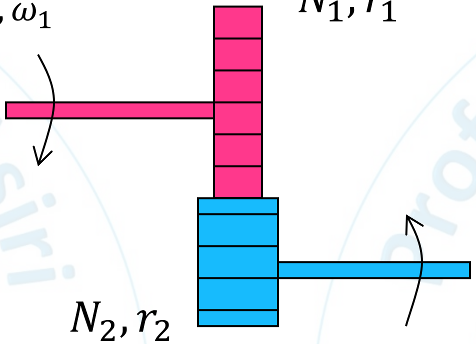
Example 4



$$\begin{cases} 3\dfrac{d^2\theta_1}{dt^2} + 2\theta_1 + 5(\theta_1 - \theta_2) + 6\dfrac{d\theta_1}{dt} = 0 \\ 4\dfrac{d^2\theta_2}{dt^2} + 5(\theta_2 - \theta_1) + 8\dfrac{d\theta_2}{dt} = 20 \end{cases}$$

Rotational Mechanical Systems

Gear train



(1) $$\frac{r_1}{r_2} = \frac{N_1}{N_2}$$

$\therefore r \propto N$

(2) $r_1\theta_1 = r_2\theta_2$

$\therefore S_1 = S_2$

(3) $\tau_1\theta_1 = \tau_2\theta_2$

no energy loss

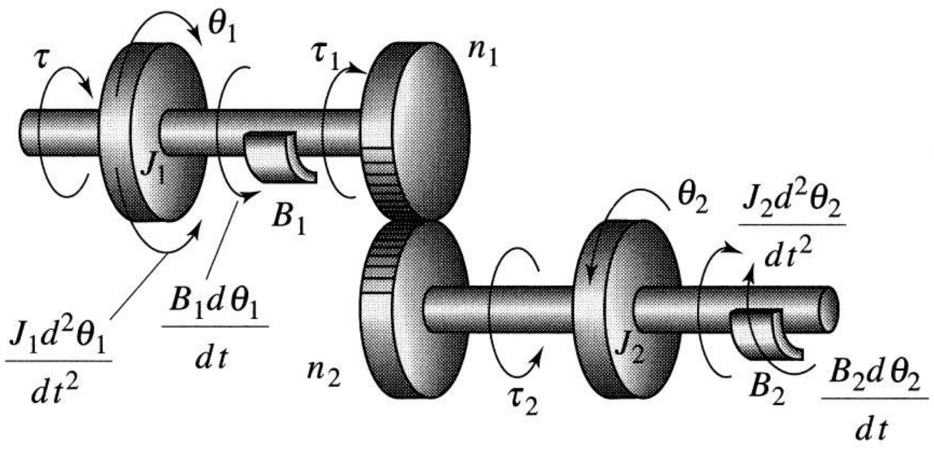
(4) $r_1\omega_1 = r_2\omega_2$

$\therefore S_1 = S_2$

$$\frac{N_1}{N_2} = \frac{r_1}{r_2} = \frac{\tau_1}{\tau_2} = \frac{\theta_2}{\theta_1} = \frac{\omega_2}{\omega_1}$$

Rotational Mechanical Systems

Example 5



(a)

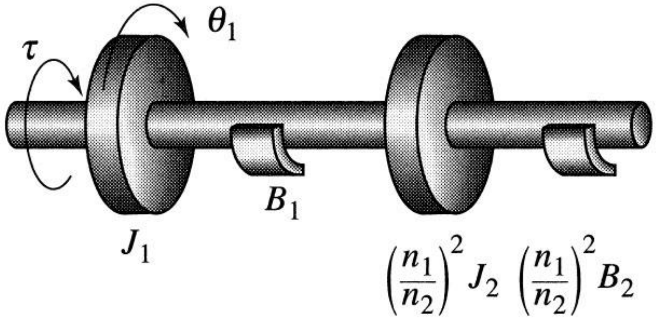
$$\tau = J_1\ddot{\theta}_1 + B_1\dot{\theta}_1 + \tau_1 \qquad (1)$$

$$\tau_2 = J_2\ddot{\theta}_2 + B_2\dot{\theta}_2 \qquad (2)$$

$$\tau_2 \Rightarrow \frac{N_2}{N_1}\tau_1$$

$$\ddot{\theta}_2 \Rightarrow \frac{N_1}{N_2}\ddot{\theta}_1$$

$$\dot{\theta}_2 \Rightarrow \frac{N_1}{N_2}\dot{\theta}_1$$



(b)

Electrical Systems

- Basic Modeling Elements
- Interconnection Relationships
- Derive Input/Output Models

Electrical Systems

Variables

- ***q*** : charge **[C] (Coulomb)**
- ***i*** : current **[A]**
- ***e*** : voltage **[V]**
- ***R*** : resistance **[Ω]**
- ***C*** : capacitance **[Farad]**
- ***L*** : inductance **[H] (Henry)**
- ***p*** : power **[Watt]**
- ***w*** : work (energy) **[J]**

  1 [J] (Joule) = 1 [V-A-sec]

$$i = \frac{dq}{dt}$$

$$q(t_1) = q(t_0) + \int_{t_0}^{t_1} i(t)\ dt$$

$$p = e.i$$

$$w(t_1) = w(t_0) + \int_{t_0}^{t_1} p(t)\ dt$$

$$= w(t_0) + \int_{t_0}^{t_1} (e \cdot i)\ dt$$

Mechanical - Electrical Systems

x	:	displacement [m]
v	:	velocity [m/sec]
f	:	force [N]
p	:	power [Nm/sec]
w	:	work (energy) [Nm]
		1 [Nm] = 1 [J] (Joule)
a	:	acceleration [m/sec^2]

$$v = \frac{dx}{dt} \qquad a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{d}{dt}x\right) = \frac{d^2}{dt^2}x = \ddot{x}$$

$$x(t_1) = x(t_0) + \int_{t_0}^{t_1} v(t)\ dt$$

$$p = f \cdot v = f \cdot \frac{dx}{dt} = \frac{dw}{dt}$$

$$w(t_1) = w(t_0) + \int_{t_0}^{t_1} p(t)\ dt$$

$$= w(t_0) + \int_{t_0}^{t_1} (f \cdot \dot{x})\ dt$$

q	:	charge [C] (Coulomb)
i	:	current [A]
e	:	voltage [V]
p	:	power [Watt]
w	:	work (energy) [J]
		1 [J] (Joule) = 1 [V-A-sec]
R	:	resistance [W]
C	:	capacitance [Farad]
L	:	inductance [H] (Henry)

$$i = \frac{dq}{dt}$$

$$q(t_1) = q(t_0) + \int_{t_0}^{t_1} i(t)\ dt$$

$$p = e.i = e.\frac{dq}{dt} = \frac{dw}{dt}$$

$$w(t_1) = w(t_0) + \int_{t_0}^{t_1} p(t)\ dt$$

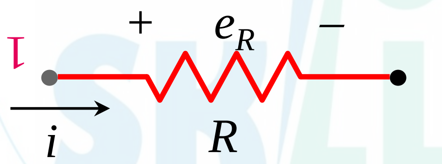
$$= w(t_0) + \int_{t_0}^{t_1} (e \cdot i)\ dt$$

Electrical Systems

Basic Modeling Elements

- **Resistor**
  - Ohms Law
    Voltage across is proportional to the through current.

$$e_{12} = e_1 - e_2 = e_R = R\ i \quad \Leftrightarrow \quad i = \frac{1}{R}\ e_R$$



Static relation

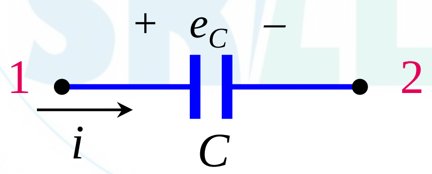
  - Dissipates energy through heat.

$$p = R\,i^2 = \frac{1}{R}\,e^2$$

  - Analogous to friction elements in mechanical systems, e.g. dampers

- **Capacitor**
  - Charge collected is proportional to the voltage across.
  - Current is proportional to the rate of change of the voltage across.

$$q = C\ e_C \quad \Leftrightarrow \quad i = C\left(\frac{d}{dt}e_C\right) = C\left(\frac{d}{dt}e_{12}\right)$$



dynamic relation

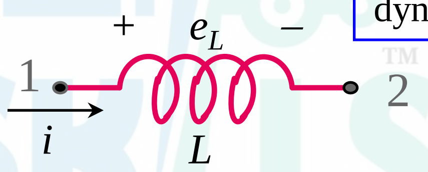
  - Energy supplied is stored in its electric field and can affect future circuit response.
  - Steady-state response: $i=0$, Open Circuit

Electrical Systems

- **Inductor**
  - Voltage across is proportional to the rate of the change of the through current.

$$e_{12} = e_L = L\left(\frac{d}{dt} i\right)$$

dynamic relation

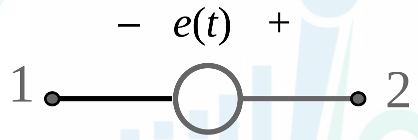


  - Energy supplied is stored in its magnetic field.

$$w = \frac{1}{2} L\, i^2$$

  - Steady-state response: *e=0*, Short Circuit

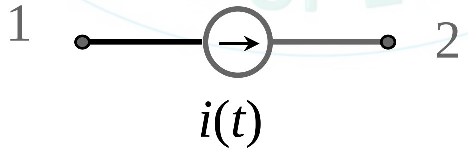
- **Voltage Source**
  - Maintain specified voltage across two points, regardless of the required current.



$$e_{12} = e_1 - e_2 = -e(t)$$

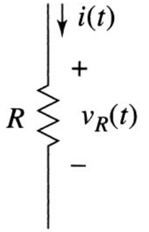
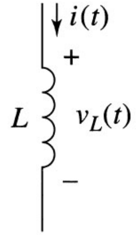
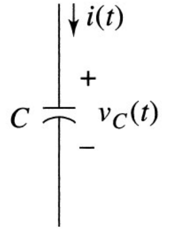
Static relation

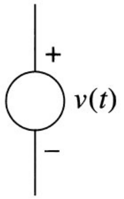
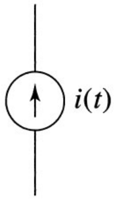
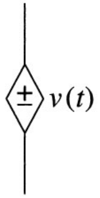
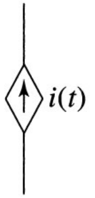
- **Current Source**
  - Maintain specified current, regardless of the required voltage.



Electrical Systems

Basic Modeling Elements

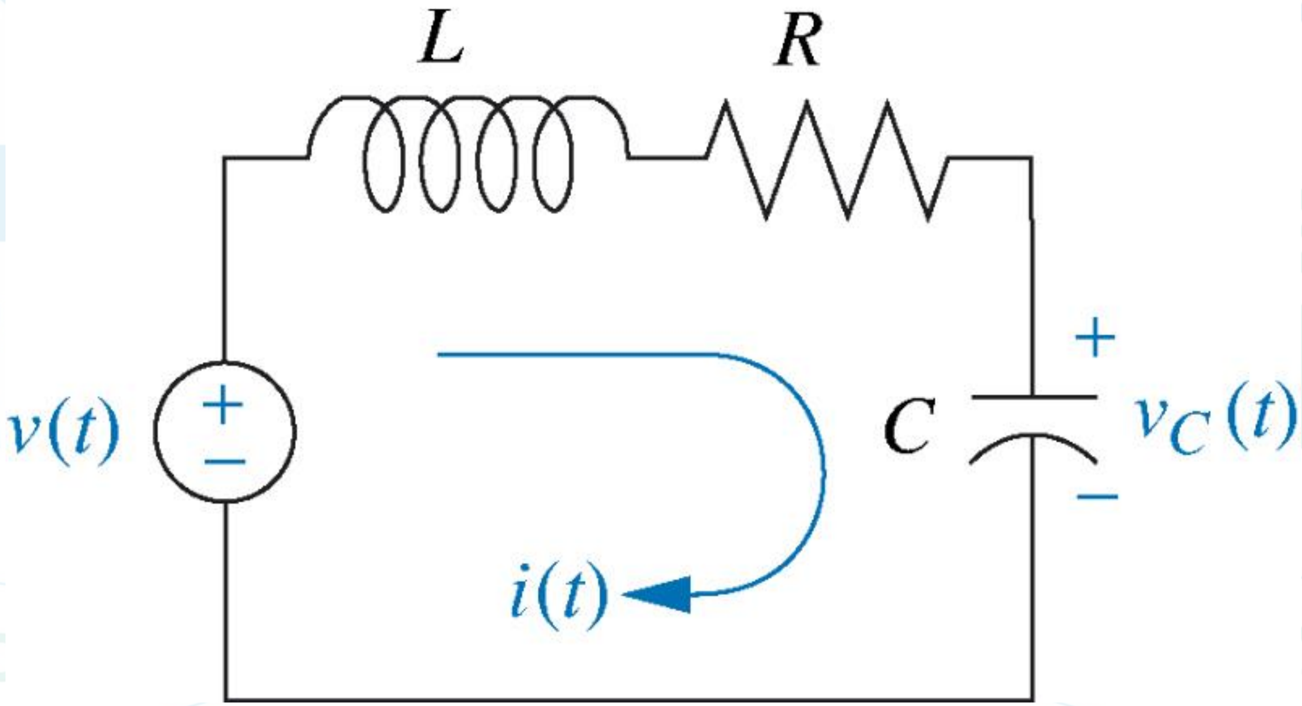
Resistor	Inductor	Capacitor
		
$v_R(t) = Ri(t)$	$v_L(t) = L\frac{di}{dt}$	$v_C(t) = \frac{1}{C}\int_{-\infty}^{t} i(t)dt$
$i(t) = \frac{1}{R}v_R(t)$	$i(t) = \frac{1}{L}\int_{-\infty}^{t} v_L(t)dt$	$i(t) = C\frac{dv_C}{dt}$

Voltage Source		Current Source	
	$v(t)$ a given function of time		$i(t)$ a given function of time
	$v(t)$ expressed in terms of other network voltages or currents		$i(t)$ expressed in terms of other network voltages or currents

Circuit Analysis – Mesh Analysis

Example 6:

Find the transfer function V_c/V.



Circuit Analysis – Mesh Analysis

Solution

Summing the voltage around the loop gives:

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int idt = v$$

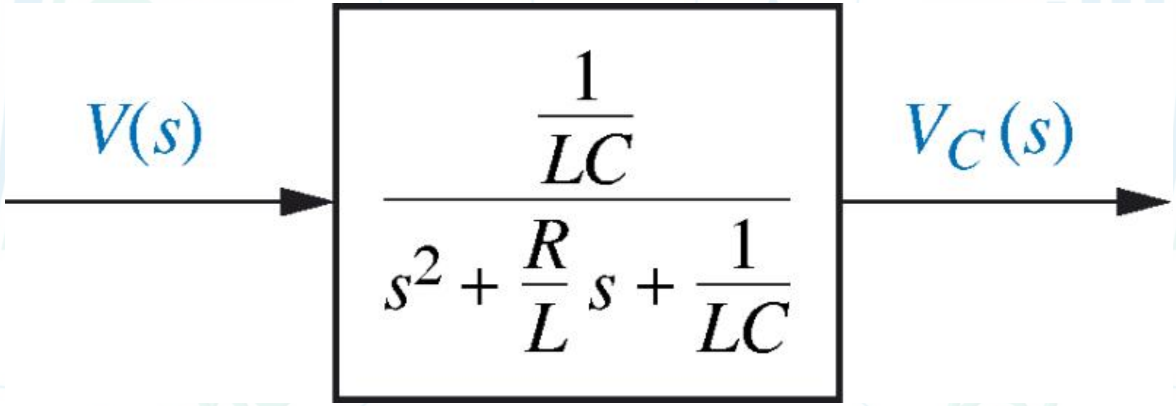
But, $i = \frac{dq}{dt}$ $\Rightarrow$ $$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = v$$

as $q = Cv_c$ $\Rightarrow$ $$LC\frac{d^2v_c}{dt^2} + RC\frac{dv_c}{dt} + v_c = v$$

Circuit Analysis – Mesh Analysis

Applying the Laplace Transform, gives

$$\boxed{\left(LCs^2 + RCs + 1\right)V_c(s) = V(s)}$$



G(s)

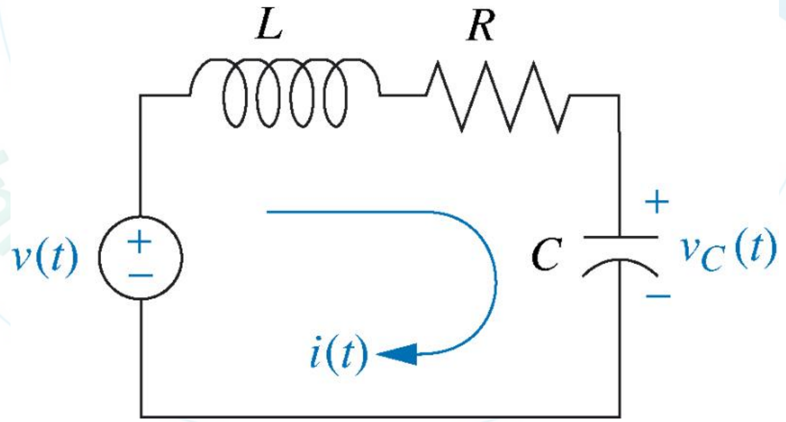
Component Transfer Function

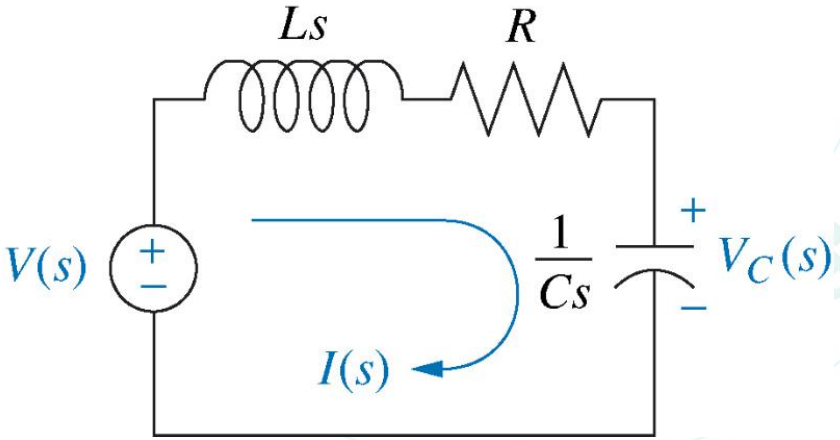
Capacitance (C) → $V(s) = \frac{1}{Cs} I(s)$

Resistance (R) → $V(s) = RI(s)$

Inductance (L) → $V(s) = LsI(s)$

$$\boxed{\frac{V(s)}{I(s)} = Z(s)}$$

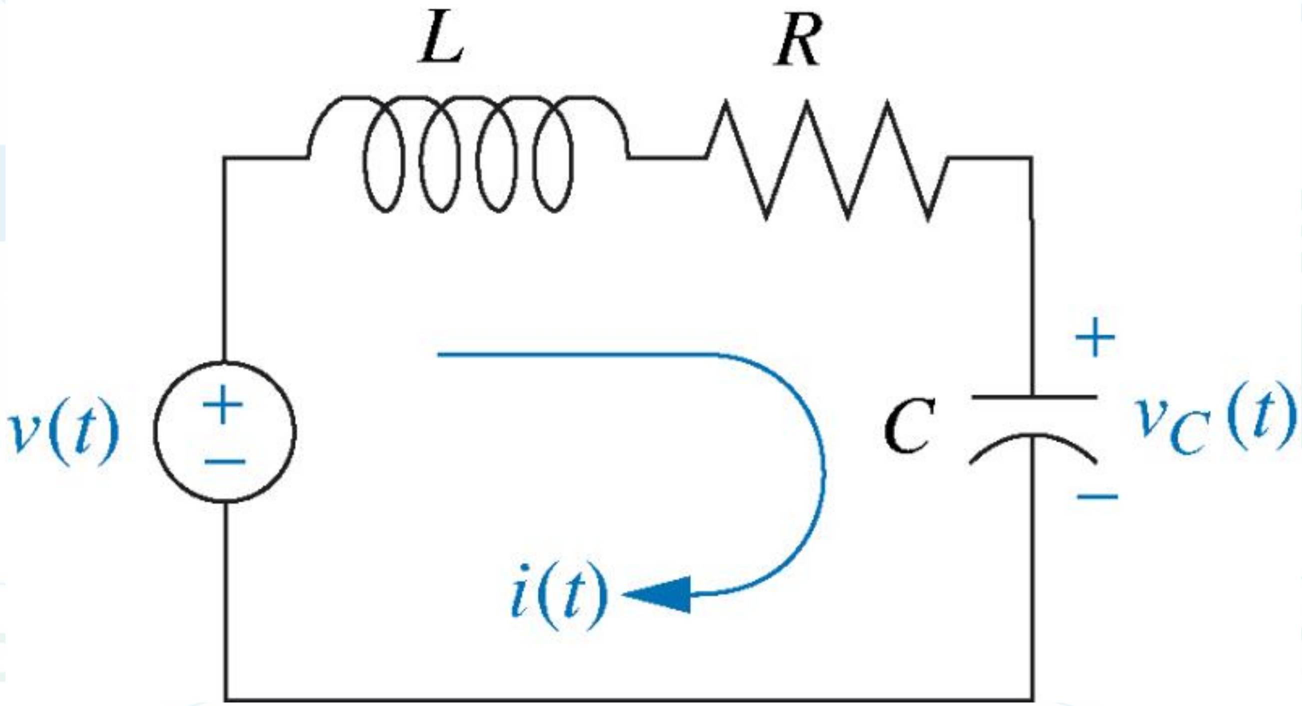




Circuit Analysis – Mesh Analysis

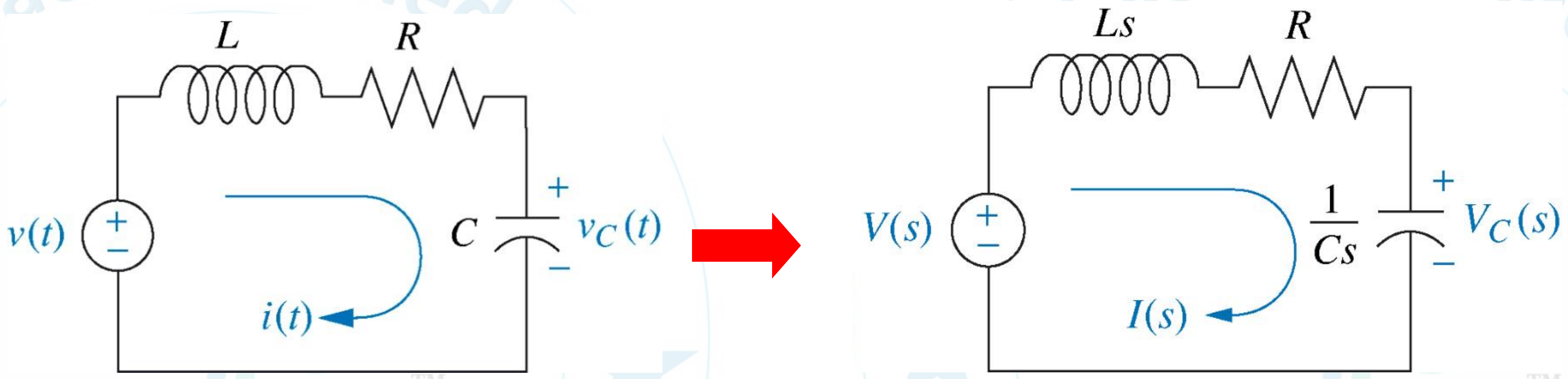
Example 7:

Find the transfer function V_c/V by mesh analysis & transform methods.



Circuit Analysis – Mesh Analysis

Solution



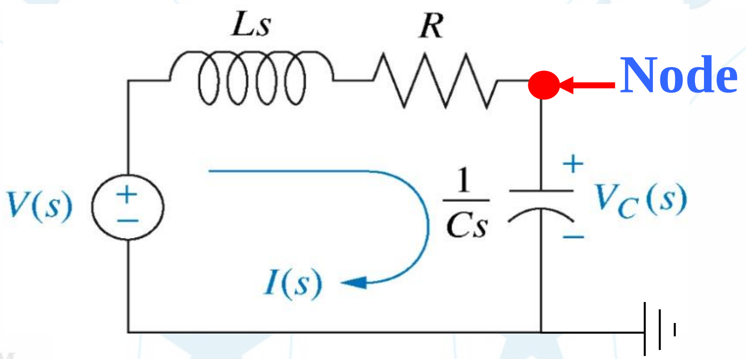
Then, as $Z(s) = Ls + R + \frac{1}{Cs}$, $\frac{V(s)}{I(s)} = Z(s)$ **&** $\frac{V_c(s)}{I(s)} = \frac{1}{Cs}$

$$\boxed{\frac{V_c(s)}{V(s)} = \frac{1}{(LCs^2 + RCs + 1)}}$$

Circuit Analysis – Nodal Analysis

Example 8:

Find the transfer function V_c/V by nodal analysis & transform methods.



Then, as $I(s) = \dfrac{V(s)}{Z(s)}$ ➡ **Kirchoff's Current Law at node**

$$\frac{V_c(s)}{\frac{1}{Cs}} + \frac{V_c(s) - V\ \ (s)}{Ls + R} = 0$$

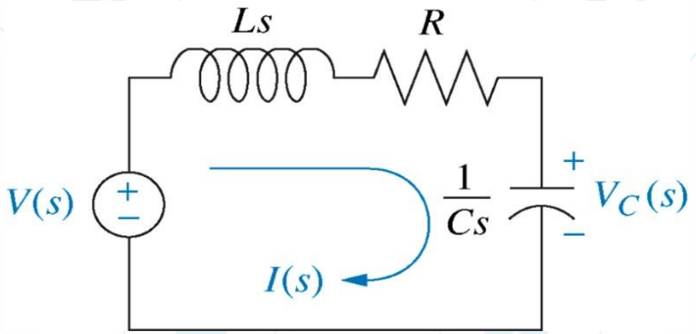
➡

$$\boxed{\frac{V_c(s)}{V(s)} = \frac{1}{(LCs^2 + RCs + 1)}}$$

Circuit Analysis – Voltage Division

Example 9:

Find the transfer function V_c/V by voltage division & transform methods.

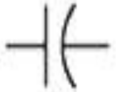


Voltage across capacitor is a portion of the input voltage which is proportional to the capacitor impedance to the sum of impedances.

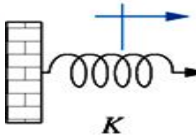
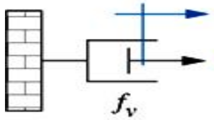
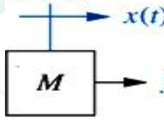
$$V_c(s) = \frac{\frac{1}{Cs}}{\left(Ls + R + \frac{1}{Cs}\right)} V(s)$$

Basic Relationship for Electrical & Mechanical Components

Electrical Components

Component	Voltage-current	Current-voltage
 Capacitor	$v(t) = \frac{1}{C}\int_0^t i(\tau)d\tau$	$i(t) = C\frac{dv(t)}{dt}$
 Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$
 Inductor	$v(t) = L\frac{di(t)}{dt}$	$i(t) = \frac{1}{L}\int_0^t v(\tau)\, d\tau$

Mechanical Components

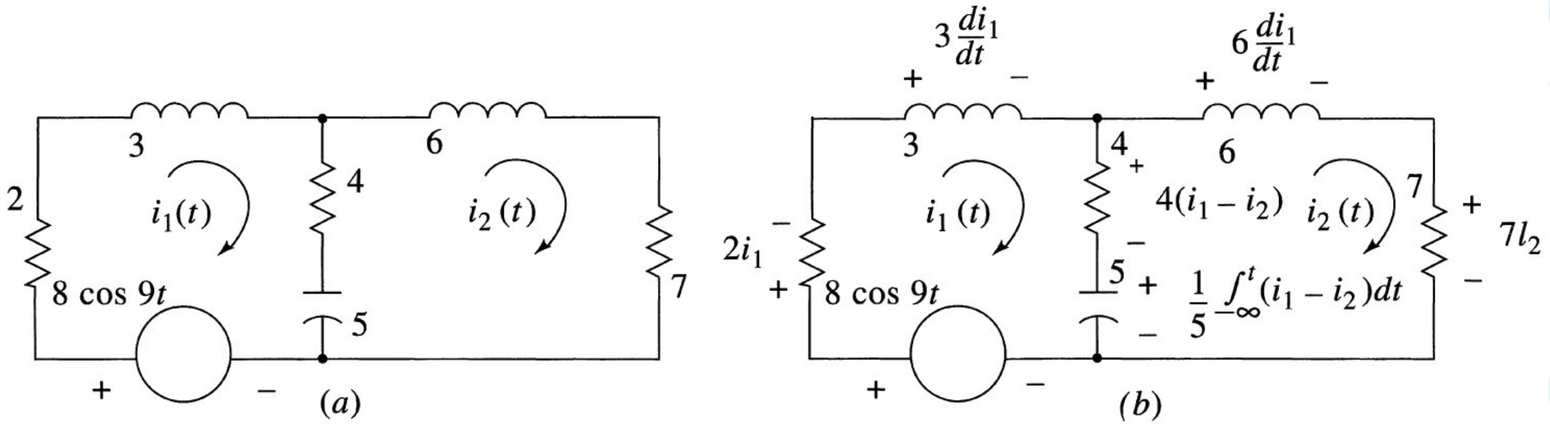
Component	Force-velocity
Spring  $x(t)$ $f(t)$ K	$f(t) = K\int_0^t v(\tau)d\tau$
Viscous damper  $x(t)$ $f(t)$ f_v	$f(t) = f_v v(t)$
Mass  $x(t)$ M $f(t)$	$f(t) = M\frac{dv(t)}{dt}$

Two analogs

- Voltage-current → Force-Velocity
- Current-voltage → Force-Velocity

Electrical Systems

Example 10:



$$loop1 \quad 2i_1(t) + 3\frac{di_1(t)}{dt} + 4(i_1(t) - i_2(t)) + \frac{1}{5}\int_0^5 (i_1(t) - i_2(t))dt = 8\cos 9\,t$$

$$loop2 \quad \frac{1}{5}\int_0^t (i_2(t) - i_1(t))dt + 4(i_2(t) - i_1(t)) + 6\frac{di_2(t)}{dt} + 7i_2(t) = 0$$

Electrical Systems

Complex impedance *Z(s)*

Ratio of *E(s)*, the Laplace transform (LP) of voltage across the terminal, to *I(s)*, the Laplace transform (LP) of current through the element, under the assumption of zero initial conditions

$$\boxed{Z(s) = \left.\frac{E(s)}{I(s)}\right|_{I.C.s=0}}$$

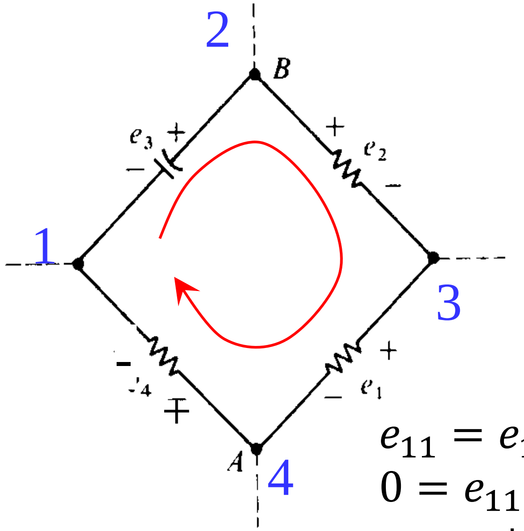
Complex impedances of resistor, capacitor, inductor are

$$R, \frac{1}{Cs}, \text{ and } Ls, \text{ respectively.}$$

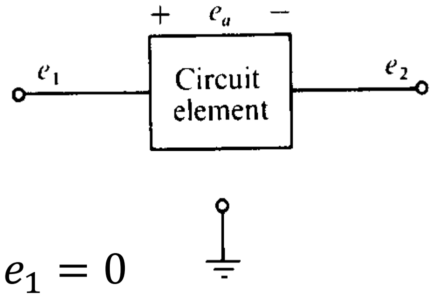
Electrical Systems

- **Kirchhoff's Voltage Law (loop law)**
  - The total voltage drop along any *closed loop* in the circuit is zero.

$$\sum_{\substack{\text{Closed}\\\text{Loop}}} e_j = 0$$



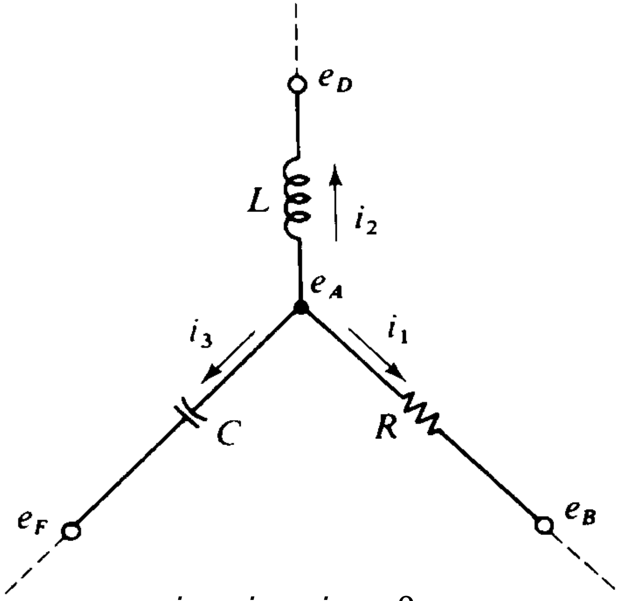
$$e_a = e_{12} = e_1 - e_2$$



$$e_{11} = e_1 - e_1 = 0$$
$$0 = e_{11}$$
$$= e_{12} + e_{23} + e_{34} + e_{41}$$
$$\underbrace{-e_3}_{e_{12}} + \underbrace{e_2}_{e_{23}} + \underbrace{e_1}_{e_{34}} + \underbrace{e_4}_{e_{41}}$$

- **Kirchhoff's Current Law (node law)**
  - The algebraic sum of the currents at any node in the circuit is zero.

$$\sum_{\substack{\text{Any}\\\text{Node}}} i_j = 0$$

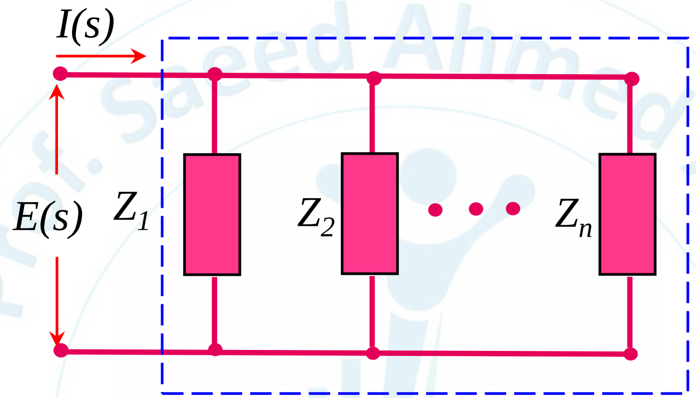


$$-i_1 - i_2 - i_3 = 0$$

Electrical Systems

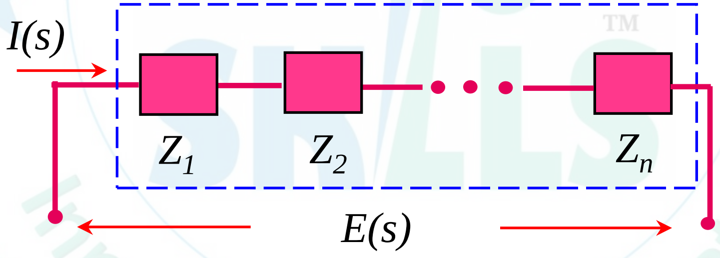
- ***Parallel combinations***

Same Voltage across elements
$\Delta(Voltage_i) = \Delta(Voltage_j)$



- ***Series combinations***

Same Current through elements
$(Current_i) = (Current_j)$



- ***Equivalent Complex Impedance***

$$\frac{1}{Z_{parallel}(s)} = \frac{1}{Z_1(s)} + \frac{1}{Z_2(s)} + \cdots + \frac{1}{Z_n(s)}$$

$$Z_{series}(s) = Z_1(s) + Z_2(s) + \cdots + Z_n(s)$$

Electrical Systems

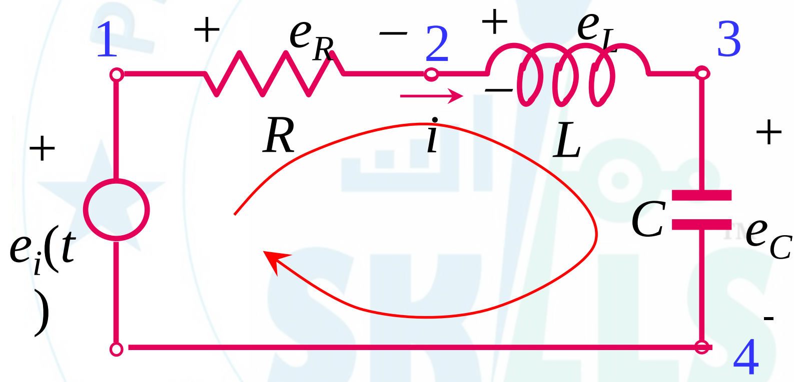
Modeling Steps

- **Understand System Function and Identify Input/Output Variables**
- **Draw Simplified Schematics Using Basic Elements**
- **Develop Mathematical Model**
  - Label Each Element and the Corresponding Voltages.
  - Label Each Node and the Corresponding Currents.
  - Apply Interconnection Laws.
  - Check that the Number of Unknown Variables equals the Number of Equations
  - Eliminate Intermediate Variables to Obtain Standard Forms.

Electrical Systems

Example 12:

Derive the I/O model for the following circuit. Let voltage $e_i(t)$ be the input and the voltage across the capacitor be the output.



Element Laws:

$$e_{12} = e_R = iR \qquad e_{23} = e_L = L\left(\frac{d}{dt}i\right)$$

$$i = C\left(\frac{d}{dt}e_{34}\right) \qquad e_{41} = -e_i(t)$$

Kirchhoff's Loop Law:

$$e_{12} + e_{23} + e_{34} + e_{41} = 0$$

No. of Unknowns:

$e_{12},\quad e_{23},\quad y = e_{34},\quad e_{41},\quad i$

Simplify

$$\begin{cases} i = C\dfrac{de_{34}}{dt} \\ iR + L\dfrac{di}{dt} + e_{34} - e_i = 0 \end{cases}$$

I/O Model:

$$\boxed{LC\frac{d^2e_{34}}{dt^2} + RC\frac{de_{34}}{dt} + e_{34} = e_i}$$

How to get I/O model by concept of complex impedance ?

Mechanical translational system

$$\boxed{M\frac{d^2x}{dt^2} + B\frac{dx}{dt} + Kx = f}$$

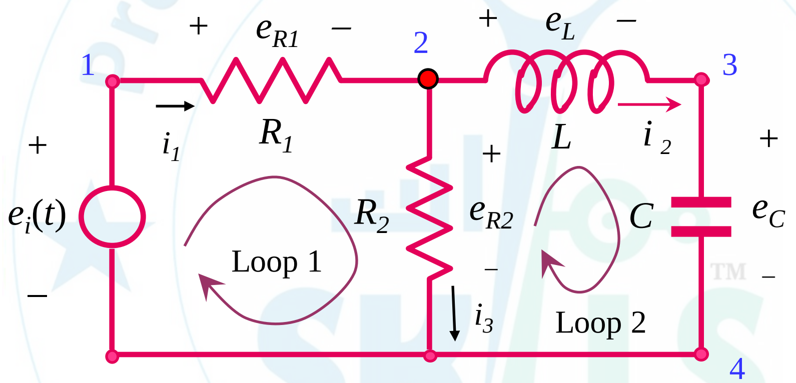
Mechanical rotational system

$$\boxed{I_c\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} + K\theta = \tau}$$

Electrical Systems

Example 13:

Obtain the I/O model for the following circuit. The input is the voltage $e_i(t)$ of the voltage source and the through current of the inductor is the output.



Elemental Equations:

$$e_{12} = e_{R1} = i_1 R_1, \qquad e_{23} = e_L = L\left(\frac{d}{dt} i_2\right)$$

$$i_2 = C\left(\frac{d}{dt} e_{34}\right), \qquad e_{24} = -e_{42} = e_{R2} = i_3 R_2$$

$$e_{41} = -e_i(t)$$

Voltage Law

Loop 1: $e_{12} + e_{24} + e_{41} = 0$

Loop 2: $e_{23} + e_{34} + e_{42} = 0$

Current Law

Node 2: $i_1 - i_2 - i_3 = 0$

Unknown Variables

e_{12}, e_{24}, e_{41}, e_{23}, e_{34}, e_{42},

i_1, $y = i_2$, i_3.

I/O Model

$$\boxed{LC\frac{d^2 i_2}{dt^2} + \frac{R_1 R_2 C}{R_1 + R_2}\frac{di_2}{dt} + i_2 = \frac{R_2 C}{R_1 + R_2}\frac{de_i}{dt}}$$

Electrical Systems

Example 13:

$$\begin{cases} i_2 = C\dfrac{de_{34}}{dt} \\ i_1R_1 + i_3R_2 - e_i = 0 \\ L\dfrac{di_2}{dt} + e_{34} - i_3R_2 = 0 \\ i_1 = i_2 + i_3 \end{cases} \Rightarrow \begin{cases} i_2 = C\dfrac{de_{34}}{dt} \\ i_2R_1 + i_3(R_1 + R_2) - e_i = 0 \\ LC\dfrac{d^2i_2}{dt^2} + C\dfrac{de_{34}}{dt} - \dfrac{di_3}{dt}CR_2 = 0 \end{cases} \Rightarrow$$

$$\begin{cases} i_3 = \dfrac{-1}{R_1 + R_2}(i_2R_1 - e_i) \\ LC\dfrac{d^2i_2}{dt^2} + i_2 - \dfrac{di_3}{dt}CR_2 = 0 \end{cases} \Rightarrow$$

$$LC\frac{d^2i_2}{dt^2} + i_2 + CR_2\frac{d}{dt}\underbrace{\frac{1}{R_1 + R_2}(i_2R_1 - e_i)}_{i_3} = 0$$

$$\boxed{LC\frac{d^2i_2}{dt^2} + \frac{R_1R_2C}{R_1 + R_2}\frac{di_2}{dt} + i_2 = \frac{R_2C}{R_1 + R_2}\frac{de_i}{dt}}$$

Transfer F^{ns} of Electrical System

Basic relationship for electrical components

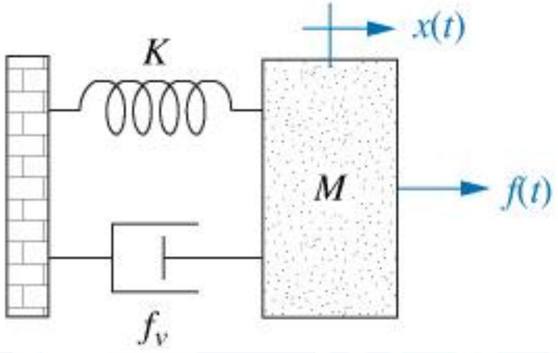
TABLE 2.3 Voltage-current, voltage-charge, and impedance relationships for capacitors, resistors, and inductors

Component	Voltage-current	Current-voltage	Voltage-charge	Impedance $Z(s)=V(s)/I(s)$	Admittance $Y(s)=I(s)/V(s)$
Capacitor	$v(t)=\frac{1}{C}\int_0^t i(\tau)d\tau$	$i(t)=C\frac{dv(t)}{dt}$	$v(t)=\frac{1}{C}q(t)$	$\frac{1}{Cs}$	Cs
Resistor	$v(t)=Ri(t)$	$i(t)=\frac{1}{R}v(t)$	$v(t)=R\frac{dq(t)}{dt}$	R	$\frac{1}{R}=G$
Inductor	$v(t)=L\frac{di(t)}{dt}$	$i(t)=\frac{1}{L}\int_0^t v(\tau)\,d\tau$	$v(t)=L\frac{d^2q(t)}{dt^2}$	Ls	$\frac{1}{Ls}$

v(t)=V(volts), i(t)=current=A (Amp), q(t)=Q(coulombs), C=F(farads), R=Ω (ohms), L=H(henries)

Electric Circuit Analogs

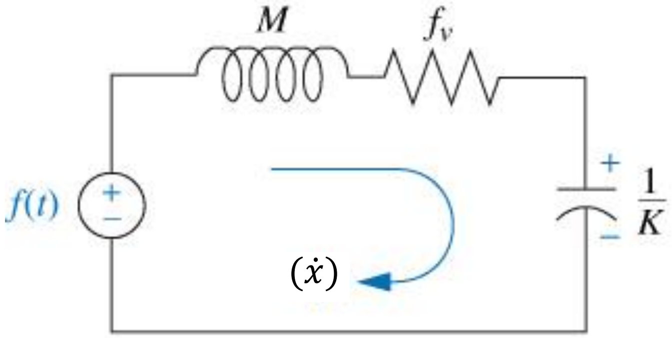
Series Analog



$$M\ddot{x} + f_v\dot{x} + Kx = f$$

Voltage (v)-current (i) → Force (f)-Velocity $(\dot{x})$

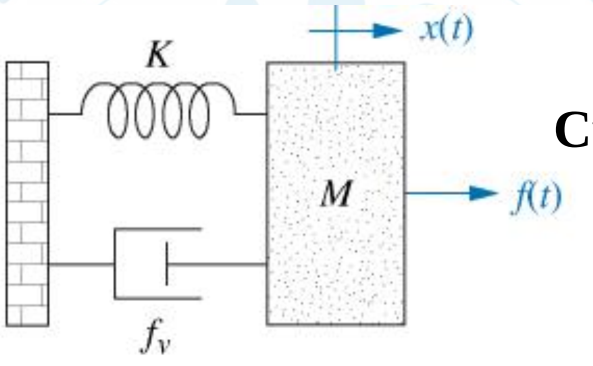
$$M\frac{di}{dt} + f_v i + K\int i dt = v$$



mass = M	→ inductor	=	M henries
viscous damper = f_v	→ resistor	=	f_v ohms
spring = K	→ capacitor	=	$\frac{1}{K}$ farads

Electric Circuit Analogs

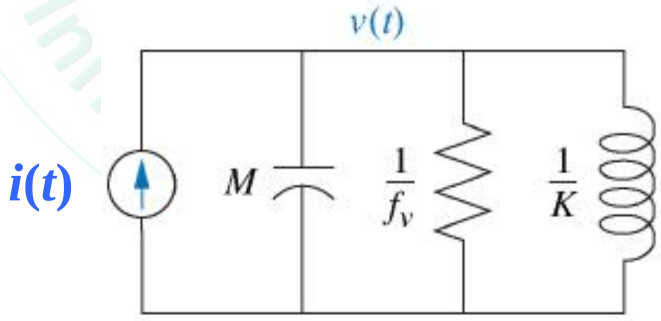
Parallel Analog



$$M\ddot{x} + f_v\dot{x} + Kx = f$$

Current (*i*) -voltage (*v*)→ Force (*f*)-Velocity $(\dot{x})$

$$M\frac{dv}{dt} + f_v v + K\int v dt = i$$

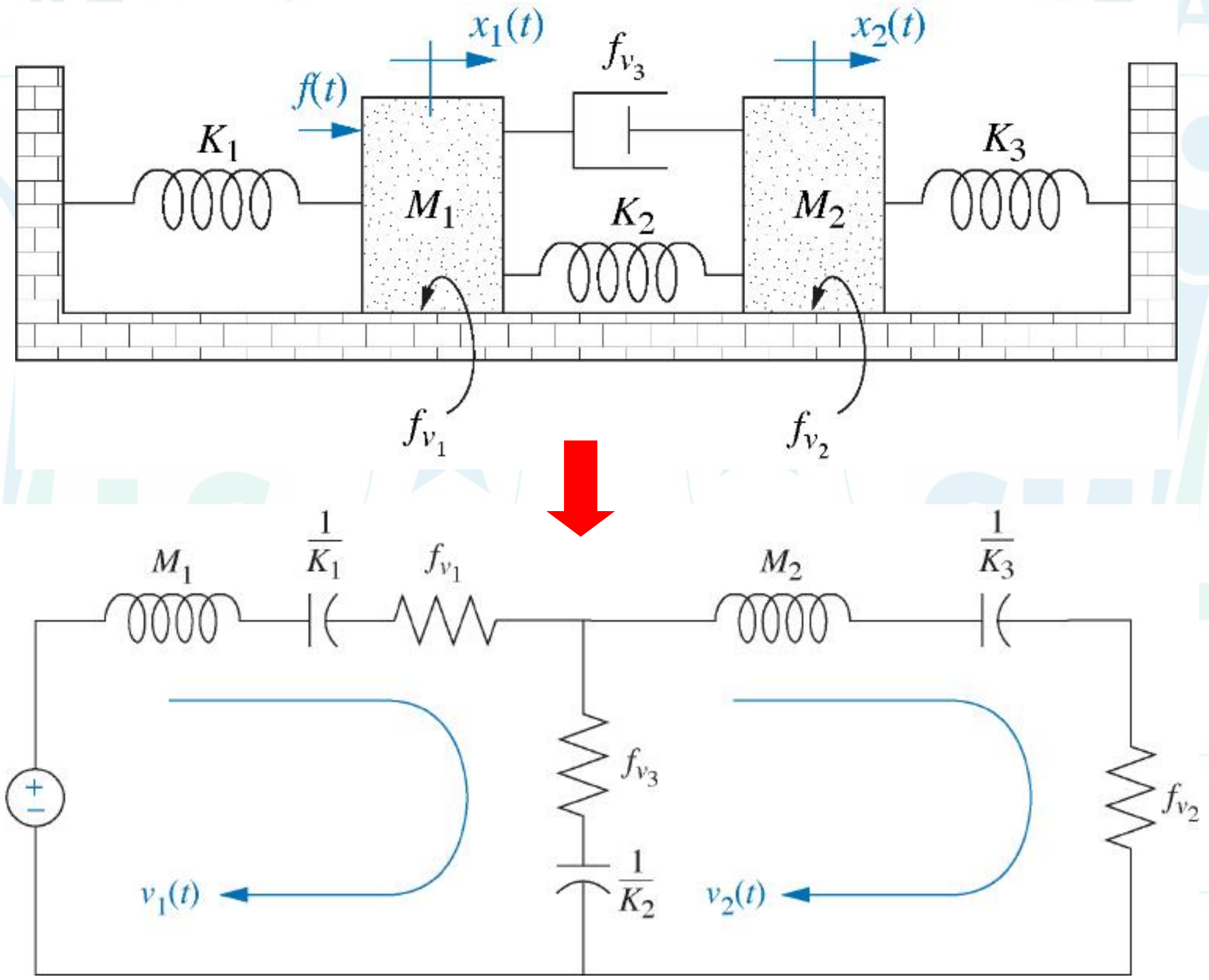


mass = M ⟶ capacitor = M farads

viscous damper = f_v ⟶ resistor = $\frac{1}{f_v}$ ohms

spring = K ⟶ inductor = $\frac{1}{K}$ henries

Electric Circuit Analogs



END