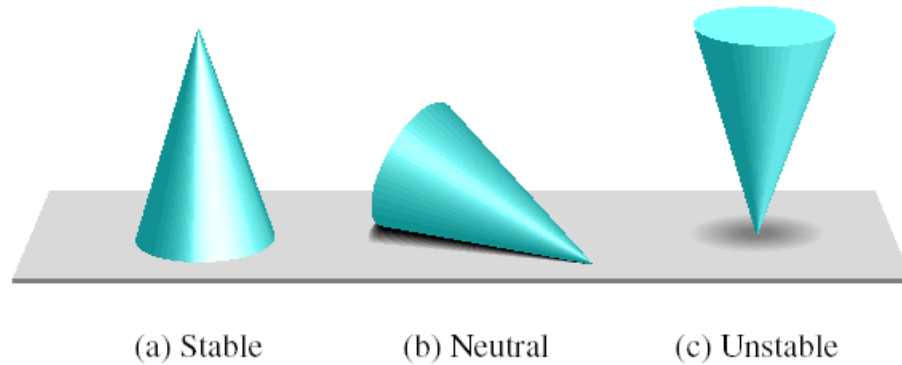


MENG366

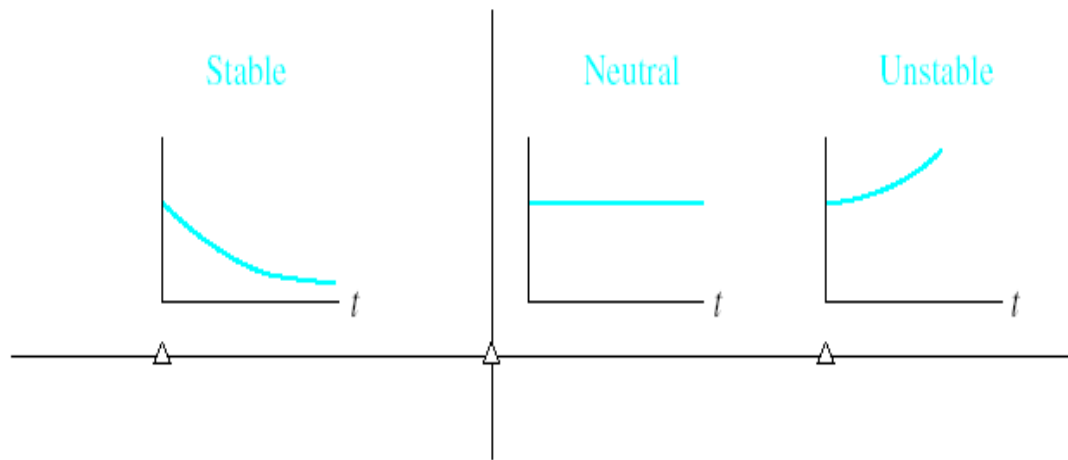
Routh's Stability Criterion

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The Concept of Stability



The concept of stability can be illustrated by a cone placed on a plane horizontal surface.



A necessary and sufficient condition for a feedback system to be stable is that all the poles of the system transfer function have negative real parts.

A system is considered marginally stable if only certain bounded inputs will result in a bounded output.

Stability

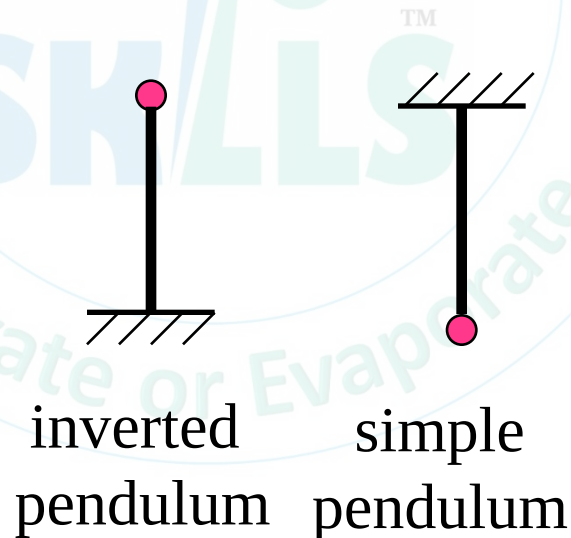
where the derivatives of all states are zeros

- Stability Concept**

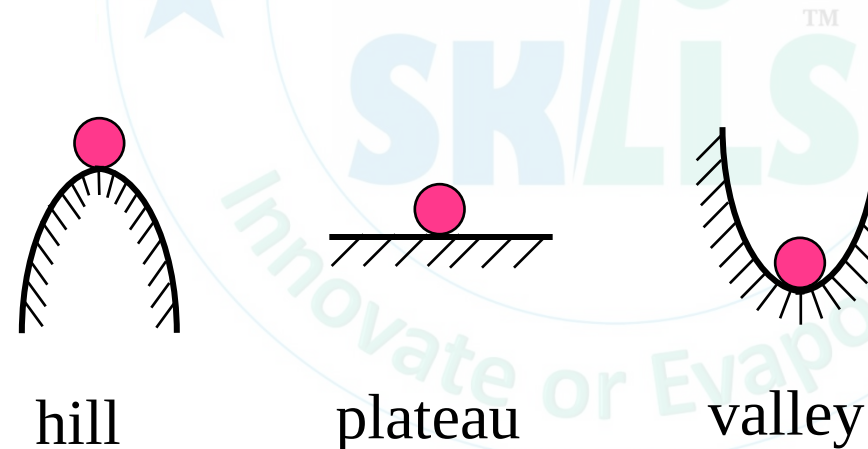
Describes the ability of a system to stay at its equilibrium position in the absence of any inputs.

- **A linear time invariant (LTI) system is stable if and only if (iff) its free response converges to zero for all ICs.**

Ex: Pendulum



Ball on curved surface

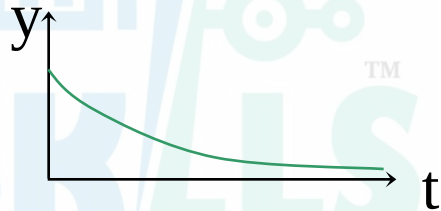


Examples (stable and unstable 1st order systems)

Q: free response of a 1st order system.

$$5\dot{y} + y = u(t) \quad y(0) = y_0$$

$$y(t) = y_0 e^{-\frac{1}{5}t}$$



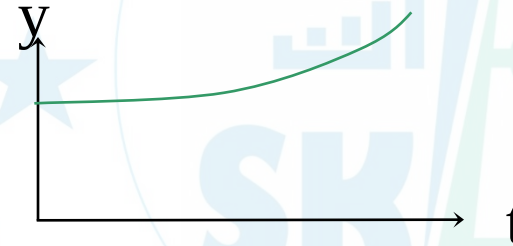
TF: $G = \frac{1}{5s+1}$

Pole: $p = -0.2$

Q: free response of a 1st order system.

$$-5\dot{y} + y = u(t) \quad y(0) = y_0$$

$$y(t) = y_0 e^{\frac{1}{5}t}$$



TF: $G = \frac{1}{-5s+1}$

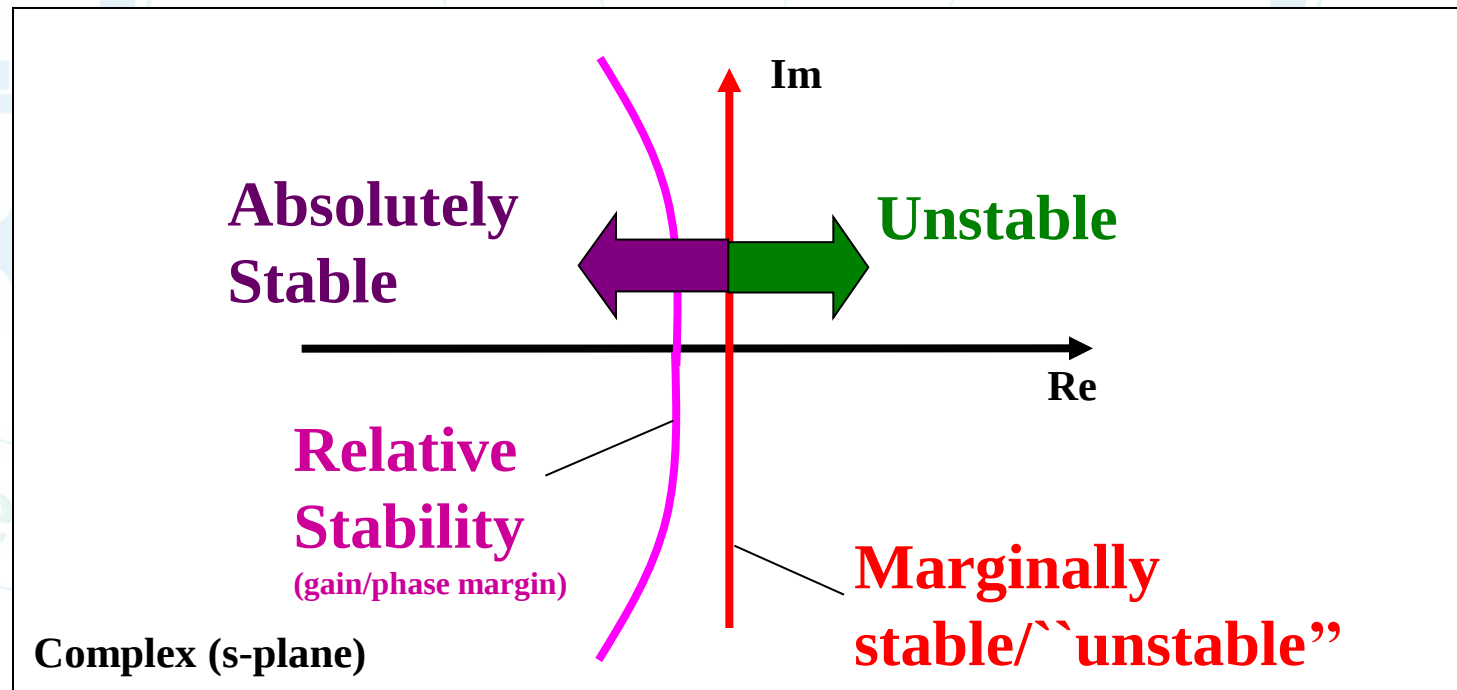
Pole: $p = 0.2$

Stability of LTI Systems

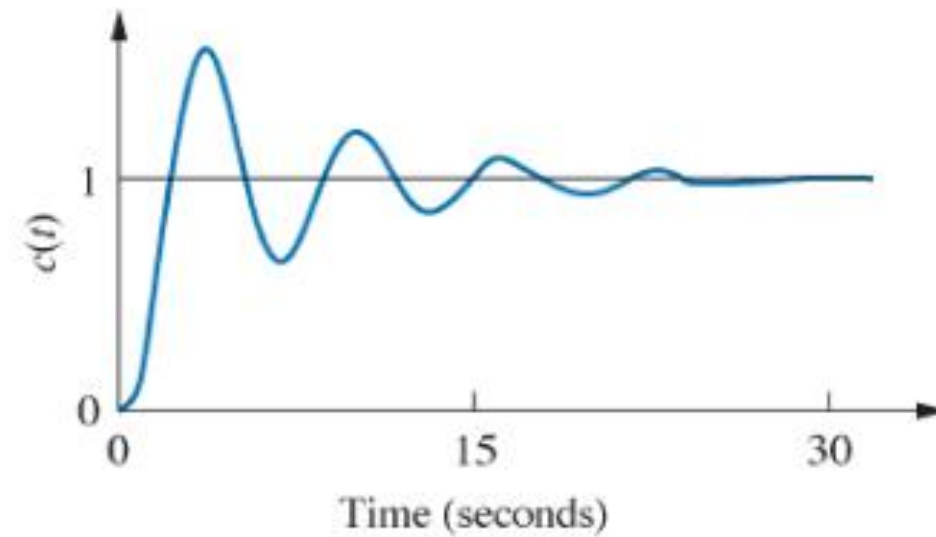
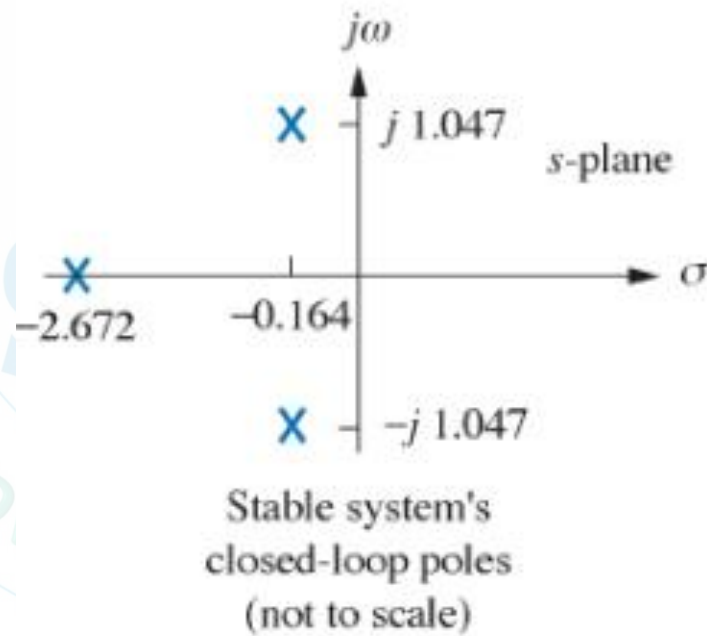
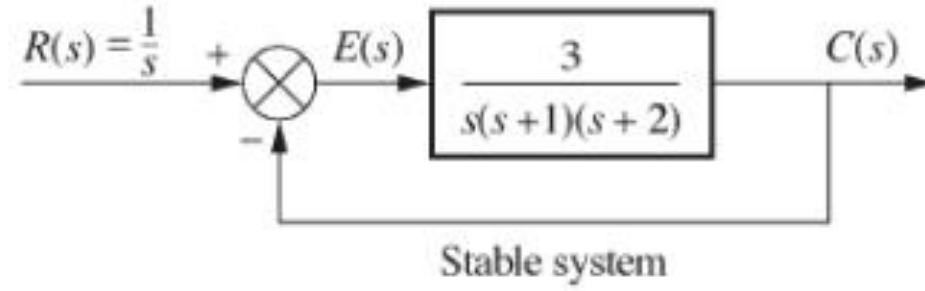
$$y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1\dot{y} + a_0y = b_mu^{(m)} + b_{m-1}u^{(m-1)} + \dots + b_1\dot{u} + b_0u$$

Stable $\iff$ All poles lie in the left-half complex plane (LHP)

$\iff$ All roots of $D(s) = \underbrace{s^n + a_{n-1}s^{n-1} + \dots + a_1s + a_0}_{\text{Characteristic Polynomial}} = 0$ lie in the LHP



Stable System



Stable System

Closed –Loop TF

$$G = \frac{3}{s^3 + 3s^2 + 2s + 3}$$

Poles

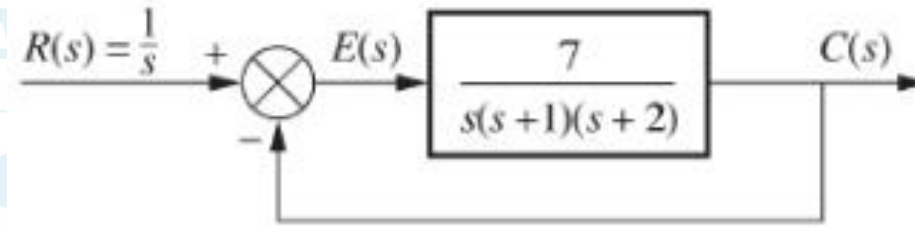
```
>> p=[1 3 2 3];  
>> roots(p)
```

ans =

```
-2.6717  
-0.1642 + 1.0469i  
-0.1642 - 1.0469i
```

➔ All in LHS of s- plane

Unstable System



Closed-Loop TF $G = \frac{7}{s^3 + 3s^2 + 2s + 7}$

Poles

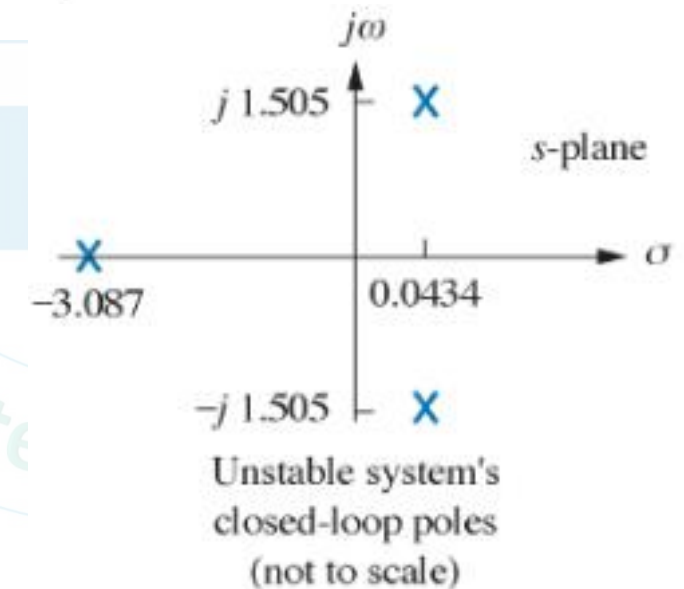
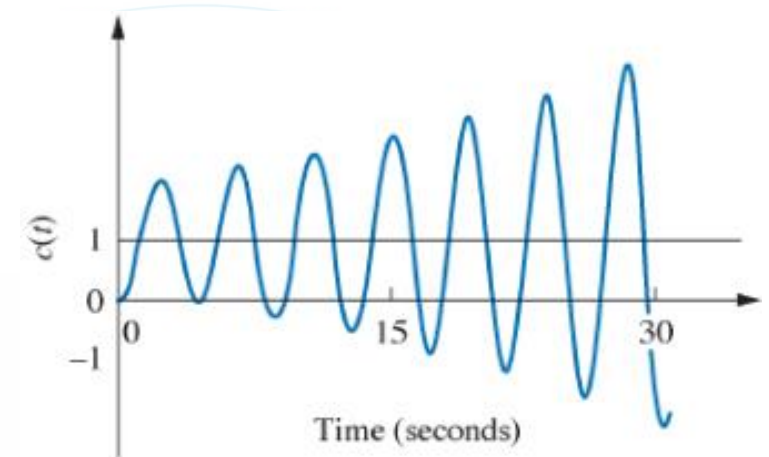
```
>> p=[1 3 2 7];
>> roots(p)
```

ans =

```
-3.0867
0.0434 + 1.5053i
0.0434 - 1.5053i
```

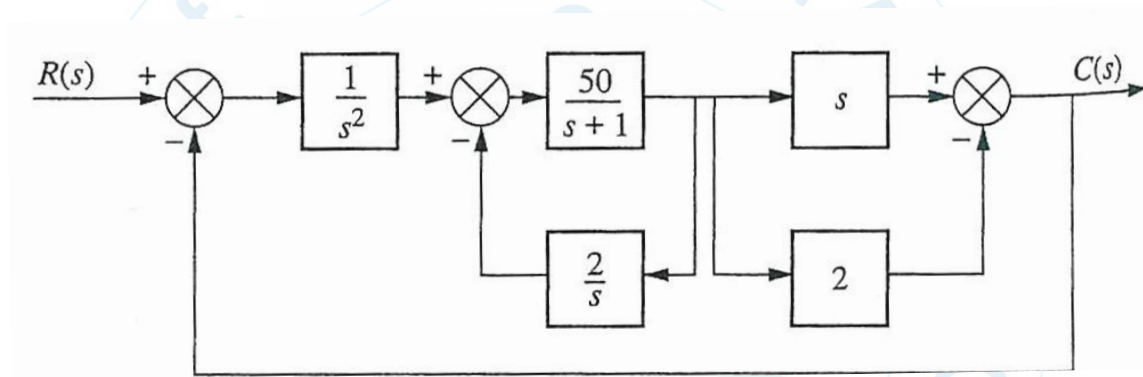


One in LHS of s- plane
Two in RHS

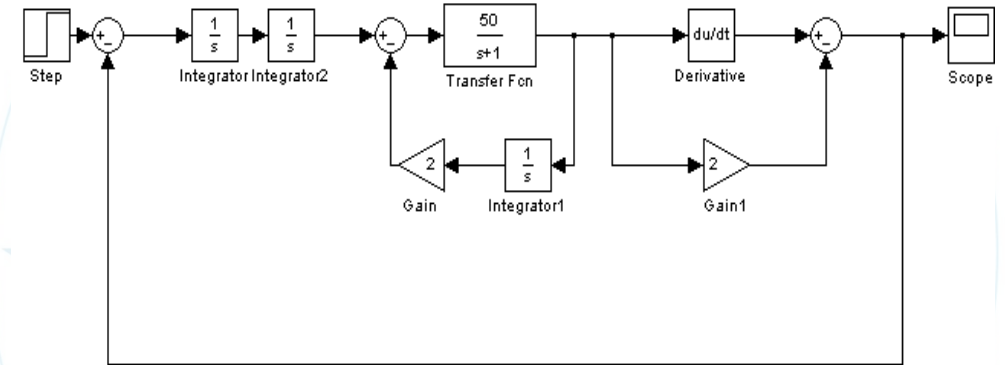


Unstable System

Block Diagram



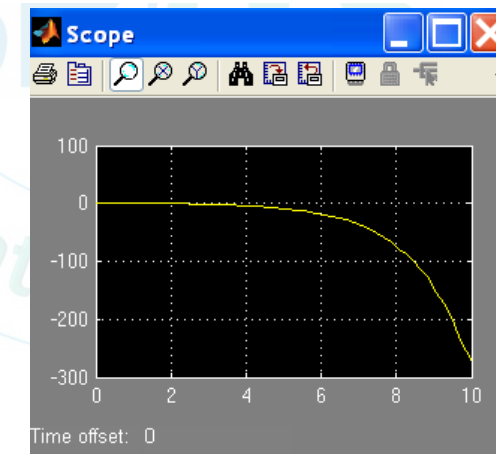
Simulink model



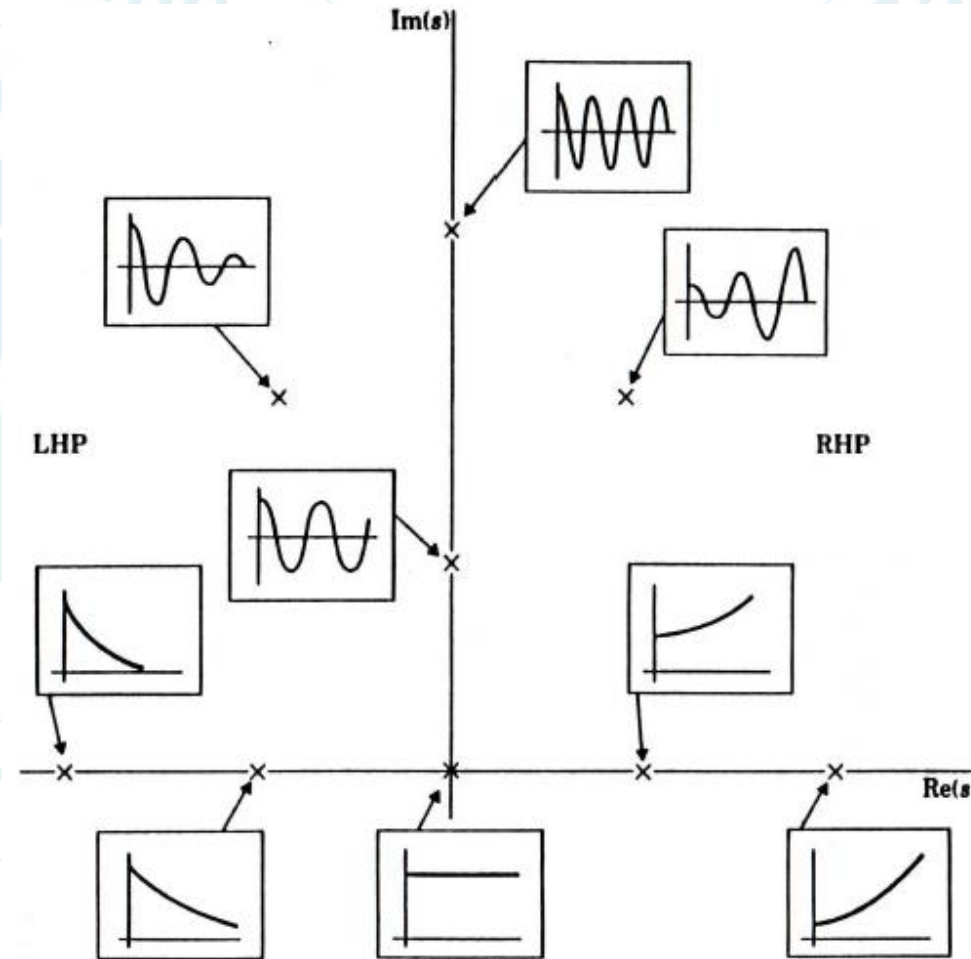
Unstable as

$$T(s) = \frac{50(s-2)}{s^3 + s^2 + 150s - 100}$$

has poles at $-0.8309 \pm 12.2642i$, 0.6618



Stable & Unstable System



STABILITY DEFINITIONS

■ Bounded Input Bounded Output Stability:

BIBO system is stable if, for every bounded input, the output remains bounded with increasing time (**all system poles must lie in the left half of the s-plane**).

■ Marginal Stability:

Occurs if some of the poles lie on the **imaginary axis**, while all others are in the LHS of the s-plane.

Internal (Asymptotic) Stable

Definition :

A system is *internal (asymptotic)* stable, if the zero-input response decays to zero, as time approaches infinity, for all possible initial conditions.

Asymptotic stable $\Rightarrow$ All the characteristic polynomial roots are located in the LHP (left-half-plan)

External (BIBO) Stable

Definition :

A system is *external (bounded-input, bounded-output, BIBO) stable*, if the zero-state response is bounded, as time approaches infinity, for all bounded inputs.

bounded-input, bounded-output stable $\Rightarrow$ All the poles of transfer function are located in the LHP (left-half-plan)

Asymptotic stable $\Rightarrow$ BIBO stable

BIBO stable $\Rightarrow$ Asymptotic stable

Poles and zeros

- Poles: values of s at which $TF \rightarrow \infty$
 - Most time, poles = roots of denominator
 - When there are common factors in numerator and denominator, cancel them first
- Zeros: values of s at which $TF = 0$
 - Finite zeros: roots of numerator
 - Number of zeros at infinity: $n-m$, $n = \text{den deg}$ and $m = \text{num deg}$
- Totally n poles and n zeros
 - n is called the order of the system
 - $n - m$ the relative order

Example

$$G(s) = \frac{10(s+1)}{s^2(s+4)(s+6)}$$

- Order: $n = 4 = \text{den deg}$
- $m = \text{num deg} = 1$
- Relative order = $n - m = 3$
- 4 poles at: 0, 0, -4, -6
- One finite zero at -1
- 3 zeros at infinity



BIBO Stability

- System is BIBO stable if any bounded input generates bounded output
- Simple criteria:
 - After common factor cancellation
 - All poles have strictly negative real parts

BIBO Stability

Exercise:

$$G_1(s) = \frac{10(s+1)}{s^2(s+4)(s+6)}$$

$$G_3(s) = \frac{s-1}{s^2+4s+6}$$

$$G_2(s) = \frac{10s+3}{(s+2)(s+5)}$$

$$G_4(s) = \frac{e^{-2s}}{10s(s+1)(s+2)}$$

BIBO Stability

Exercise:

$$G_1(s) = \frac{10(s+1)}{s^2(s+4)(s+6)}$$

$$G_3(s) = \frac{s-1}{s^2+4s+6}$$

$$G_2(s) = \frac{10s+3}{(s+2)(s+5)}$$

$$G_4(s) = \frac{e^{-2s}}{10s(s+1)(s+2)}$$

G1 and G4 are not BIBO stable

G2 and G3 are BIBO stable

STABILITY ANALYSIS

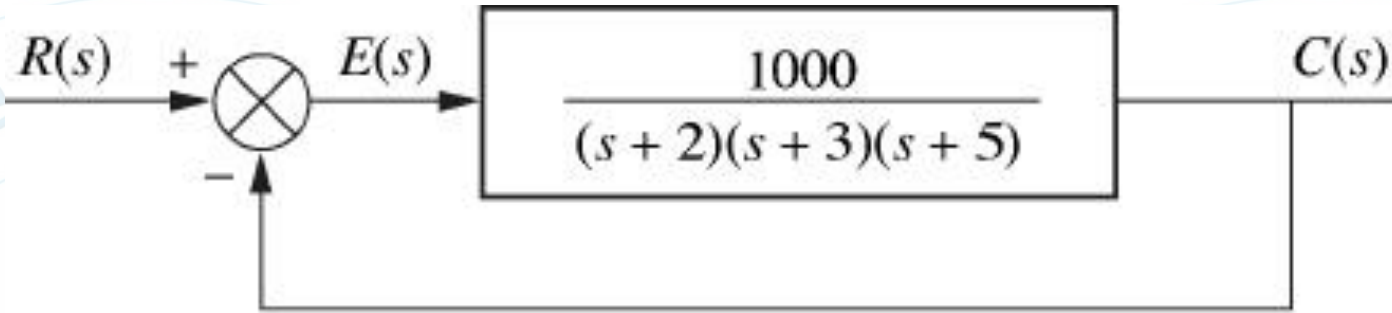
Methods for Testing Stability of a LTI system

1. Examine the poles of the system.

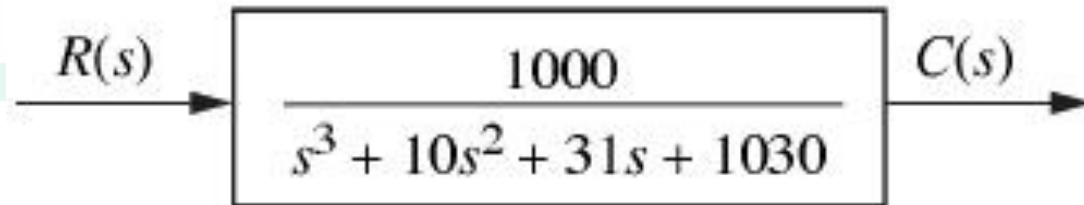
2. Methods that do not require the actual solution of the characteristic equation.

3. Methods that are based only on the loop transfer function characteristics.

Poles of Complex Systems



Open-loop poles: $s = -2, -3, -5$



Closed-loop poles: $-13.4136, 1.7068 + 8.5950j, 1.7068 - 8.5950j$

Routh-Hurwitz Criterion

Is a method that checks system stability without actual solution of the characteristic equation.

Is a method for establishing bounds on system parameters to ensure stability.

Nuoki Mutsumoto, “Simple Proof of The Routh Stability Criterion Based on Order Reduction of Polynomials and Principle of Argument”, *The 2001 IEEE International Symposium on Circuits and Systems*, 6-9 May, 2001, Sydney Australia, Vol. I, pp. 699-702

E. Routh & A. Hurwitz

Edward John Routh
(1831-1907)



University of Cambridge

Adolf Hurwitz
(1859 -1919)



Eidgenössische Polytechnikum Zürich

Routh-Hurwitz Stability Criterion

A quick method for checking BIBO stability

- Assume the characteristic polynomial is

$$Q(s) = a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0$$

where $a_0 \neq 0$.

- A **necessary** (but not sufficient) condition for all roots to have non-positive real parts is that **all coefficients have the same sign**.
- For the **necessary and sufficient** conditions, the sign of the first column of the **Routh array** should not change.

The Routh Array

$$Q(s) = a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0$$

s^n	a_n	a_{n-2}	a_{n-4}	a_{n-6}	...
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	a_{n-7}	...
s^{n-2}	b_1	b_2	b_3	b_4	...
s^{n-3}	c_1	c_2	c_3	c_4	...
$\vdots$	$\vdots$	$\vdots$	$\vdots$		
s^2	k_1	k_2			
s^1	l_1				
s^0	m_1				

$$b_1 = \frac{a_{n-1} a_{n-2} - a_n a_{n-3}}{a_{n-1}}$$

$$b_2 = \frac{a_{n-1} a_{n-4} - a_n a_{n-5}}{a_{n-1}}$$

etc.

$$c_1 = \frac{b_1 a_{n-3} - a_{n-1} b_2}{b_1}$$

$$c_2 = \frac{b_1 a_{n-5} - a_{n-1} b_3}{b_1}$$

Routh-Hurwitz Stability Criterion

Necessary and sufficient conditions:

- If all elements in the **first column** of the Routh array have the **same sign**, then all roots of the characteristic equation have negative real parts.
- If there are sign changes in these elements, then the number of roots with non-negative real parts is equal to the **number of sign changes**.
- Elements in the first column which are **zero** define a special case.

Proof of Routh-Hurwitz Criterion

Second Order Systems

$$Q(s) = s^2 + as + b = 0$$

with a, b real & positive

Routh Table

s^2	1	b
s^1	a	0
s^0	b_1	

$$b_1 = \frac{a \times b - 1 \times 0}{a} = b$$

Necessary Condition

System is stable if a & b are positive, i.e. no sign changes

Proof of Routh-Hurwitz Criterion

Note that the roots of the CE are:

$$s_{1,2} = \frac{-a \pm \sqrt{-a^2 + 4b}j}{2}$$



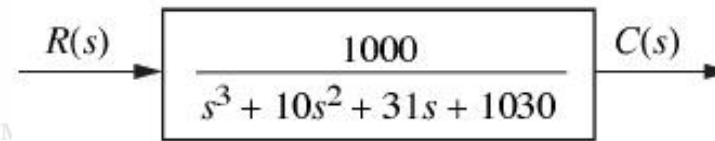
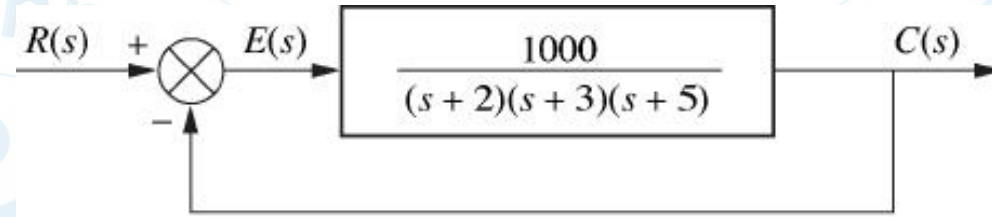
$$s_1 + s_2 = -a, \quad s_1 s_2 = b$$

Sufficient Condition

Or a & b are positive only when the roots have negative real parts

Routh-Hurwitz Stability Criterion

Example:



s^3	1	31	0
s^2	10	1030	0
s^1	$-\begin{vmatrix} 1 & 31 \\ 1 & 103 \end{vmatrix} = -72$	$-\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0$	$-\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} = 0$
s^0	$-\begin{vmatrix} 1 & 103 \\ -72 & 0 \end{vmatrix} = 103$	$-\begin{vmatrix} 1 & 0 \\ -72 & 0 \end{vmatrix} = 0$	$-\begin{vmatrix} 1 & 0 \\ -72 & 0 \end{vmatrix} = 0$

Routh-Hurwitz Stability Criterion

Example:

s^3	1	31	0	0
s^2	10	1030	0	
s^1	-72	0		
s^0	1030			

Two sign changes → two roots with positive real parts

```
>> p=[1 10 31 1030];  
>> roots(p)  
  
ans =  
-13.4136  
1.7068 + 8.5950i  
1.7068 - 8.5950i
```

Routh-Hurwitz Stability Criterion

Example:

$$Q(s) = 2s^4 + s^3 + 3s^2 + 5s + 10$$

s^4	2	3	10	0
s^3	1	5	0	0
s^2	b_1	b_2	0	
s^1	c_1	0		
s^0	d_1			

$$b_1 = \frac{3-10}{1} = -7 \quad b_2 = \frac{10-0}{1} = 10$$

$$c_1 = \frac{-35-10}{-7} = 6.43$$

$$d_1 = \frac{10(6.43)-0}{6.43} = 10$$

The characteristic equation has **two** roots with **positive real parts** since the elements of the first column have two sign changes. (**2,1,-7,6.43,10**)

Routh-Hurwitz Stability Criterion

Roots

$$Q(s) = 2s^4 + s^3 + 3s^2 + 5s + 10$$

```
>> p=[2 3 5 10];  
>> roots(p)
```

ans =

```
-1.7279  
0.1139 + 1.6973i  
0.1139 - 1.6973i
```

The characteristic equation has **two** roots with **positive real parts** since the elements of the first column have two sign changes. (**2,1,-7,6.43,10**)

Routh-Hurwitz Stability Criterion

Special Case 1

- A **zero** in the first column:
- Remedy: substitute **ε** for the zero element, finish the Routh array, and then let **$\varepsilon \rightarrow 0$** .

$$Q(s) = s^3 - 3s + 2$$

s^3	1	-3	0	0
s^2	$0(\varepsilon)$	2	0	
s^1	b_1	0		
s^0	c_1			

$$b_1 = \frac{-3\varepsilon - 2}{\varepsilon} \rightarrow \frac{-2}{\varepsilon} \text{ (negative)}$$

$$c_1 = \frac{b_1 \times 2}{b_1} = 2$$

There are **two** roots with positive real parts (**1, ε , $-2/\varepsilon$, 2**)

Routh-Hurwitz Stability Criterion

Special Case 1

$$Q(s) = s^3 - 3s + 2$$

```
>> p=[1 0 -3 2];  
>> roots(p)
```

```
ans =  
-2.0000  
1.0000  
1.0000
```

There are **two** roots with positive real parts (**1, ε , $-2/\varepsilon$, 2**)

Example on Routh Stability

Determine the stability of the closed-loop transfer function

$$T(s) = \frac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3}$$

Example on Routh Stability

Routh Table

s^5	1	3	5
s^4	2	6	3
s^3	$\emptyset \quad \epsilon$	$\frac{7}{2}$	0
s^2	$\frac{6\epsilon - 7}{\epsilon}$	3	0
s^1	$\frac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$	0	0
s^0	3	0	0

Example on Routh Stability

Routh Table

Label	First column	$\epsilon = +$	$\epsilon = -$
s^5	1	+	+
s^4	2	+	+
s^3	$\emptyset \quad \epsilon$	+	-
s^2	$\frac{6\epsilon - 7}{\epsilon}$	-	+
s^1	$\frac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$	+	+
s^0	3	+	+

Interpretation of Results

Two sign changes $\rightarrow$ two roots with positive real parts

```
>> p=[1 2 3 6 5 3];
>> roots(p)
```

ans =

0.3429 + 1.5083i

0.3429 - 1.5083i

-1.6681

-0.5088 + 0.7020i

-0.5088 - 0.7020i

Special Case 1 - Another Method

As the characteristics equation is

$$s^n + a_{n-1}s^{n-1} + \dots + a_1s + a_0 = 0$$

Let, $s = 1/d$

$$\rightarrow \left(\frac{1}{d}\right)^n + a_{n-1}\left(\frac{1}{d}\right)^{n-1} + \dots + a_1\left(\frac{1}{d}\right) + a_0 = 0$$

Then, d are reciprocal of the roots s

$$\rightarrow \left(\frac{1}{d}\right)^n [1 + a_{n-1}d + \dots + a_1d^{n-1} + a_0d^n] = 0$$

Example on Routh Stability

Determine the stability of the closed-loop transfer function

$$T(s) = \frac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3}$$

➔ Characteristics Eqⁿ:

$$3d^5 + 5d^4 + 6d^3 + 3d^2 + 2d + 1 = 0$$

Example on Routh Stability

d^5	3	6	2
d^4	5	3	1
d^3	4.2	1.4	
d^2	1.33	1	
d^1	-1.75		
d^0	1		

Two sign changes → two roots with positive real parts

No Zeros in first column

Routh-Hurwitz Stability Criterion

Special Case 2

- An **all zero row** in the Routh array which corresponds to pairs of roots with opposite signs.
- Remedy:
 - form an **auxiliary polynomial** from the coefficients in the row above.
 - Replace the zero coefficients from the coefficients of the **differentiated auxiliary polynomial**.
 - If there is no sign change, the roots of the auxiliary equation define the roots of the system on the imaginary axis.

Meaning of Entire Row of Zeros

Special Case 2 - Example

For purely even or odd polynomials, as

$$s^4 + 5s^2 + 6 = 0$$

→ Row of zeros as:

s^4	1	5	6	0
s^3	0	0	0	0
s^2				
s^1				
s^0				



s^4	1	5	6	0
s^3	4	10	0	
s^2	2.5	6	0	
s^1	0.4			
s^0	6			

Routh-Hurwitz Stability Criterion

Parameter Range Test

- The Routh-Hurwitz stability criterion may be used to find the **range of a parameter** for which the closed-loop systems is stable.
-
- Leave the parameter as an unknown coefficient in the characteristic polynomial, form the Routh array, check the range of the parameter such that the first column does not change sign.

Routh-Hurwitz Stability Criterion

Parameter Range Example

$$Q(s) = s^4 + 6s^3 + 11s^2 + 6s + K$$

s^4	1	11	K	0
s^3	6	6	0	0
s^2	10	K	0	
s^1	c_1	0		
s^0	d_1			

$$c_1 = \frac{60 - 6K}{10} \quad d_1 = K$$

Stability requires:

$$K > 0$$

$$60 - 6K > 0 \Rightarrow K < 10$$

$$\therefore 0 < K < 10$$

Routh-Hurwitz Stability Criterion

Determine the stability of the closed-loop transfer function

$$T(s) = \frac{10}{s^5 + 7s^4 + 6s^3 + 42s^2 + 8s + 56}$$

Routh-Hurwitz Stability Criterion

s^5		1		6		8
s^4	7	1		42	6	56
s^3	0	4	1	0	12	3
s^2		3		8		0
s^1		$\frac{1}{3}$		0		0
s^0		8		0		0

$$P(s) = s^4 + 6s^2 + 8$$

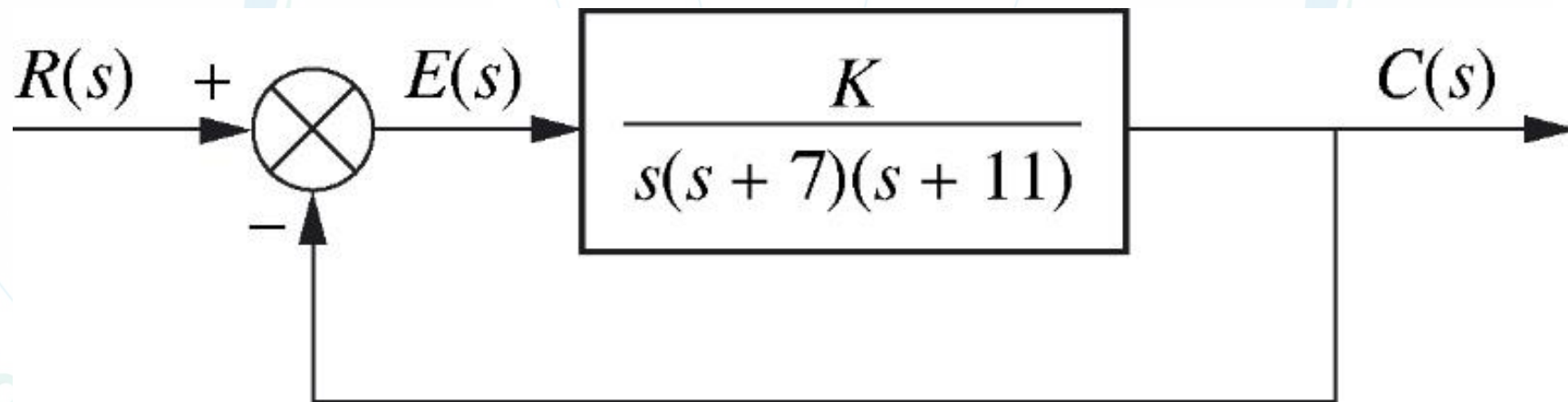
$$\Downarrow$$

$$\frac{dP(s)}{ds} = 3s^3 + 12s + 0 \Rightarrow s^3 + 4s + 0$$

**Marginally
Stable System**

Routh-Hurwitz Stability Criterion

Find the range of K , for which the following system will be stable, marginally stable, & unstable. Assume $K > 0$.



Routh-Hurwitz Stability Criterion

s^3	1	77
s^2	18	K
s^1	$\frac{1386 - K}{18}$	
s^0	K	

$K < 1386 \rightarrow$ Stable

$K > 1386 \rightarrow$ Unstable

Routh-Hurwitz Stability Criterion

$$K=1386$$

s^3	1	77
s^2	18	1386
s^1	$\emptyset$ 36	
s^0	1386	

Since there is no sign change $\rightarrow$ system is marginally stable

Routh-Hurwitz Stability Criterion

$K=1386$



$$\frac{C}{R} = \frac{1386}{s^3 + 18s^2 + 77s + 1386}$$

```
>> p=[1 18 77 1386];
```

```
>> roots(p)
```

```
ans =
```

```
-18.0000
```

```
0 + 8.7750i
```

```
0 - 8.7750i
```

Since there is no sign change → system is marginally stable

A necessary condition for Routh Stability

- A *necessary condition for stability* of the system is that all of the roots of its characteristic equation have negative real parts, which in turn requires that ***all the coefficients be positive.***

A necessary (but not sufficient) condition for stability is that ***all the coefficients of the polynomial characteristic equation be positive.***

A necessary and sufficient condition for Stability

- Routh's formulation requires the computation of a triangular array that is a function of the coefficients of the polynomial characteristic equation.

A system is stable if and only if *all* the elements of the first column of the Routh array are positive

Characteristic Equation

- Consider an nth-order system whose the characteristic equation (*which is also the denominator of the transfer function*) is:

$$a(s) = s^n + a_1s^{n-1} + a_2s^{n-2} + \dots + a_{n-1}s^1 + a_0s^0$$

Stability testing

Properties of the polynomial coefficients :

- Differing algebraic signs

$$7s^6 + 5s^4 - 3s^3 - 2s^2 + s + 10$$

At least one RHP root

- Zero-valued coefficients

$$s^6 + 3s^5 + 2s^4 + 8s^2 + 3s + 17$$

Has imaginary axis roots or RHP roots or both

- All of the same algebraic sign, non zero

$$8s^5 + 6s^4 + 3s^3 + 2s^2 + 7s + 10$$

No direct information

Method for determining the Routh array

- Consider the characteristic equation:

$$a(s) = 1 \times s^n + a_1 s^{n-1} + a_2 s^{n-2} + a_3 s^{n-3} + \dots + a_{n-1} s^1 + a_0 s^0$$

- First arrange the coefficients of the characteristic equation in two rows, beginning with the first and second coefficients and followed by the even-numbered and odd-numbered coefficients:

$$\begin{array}{ccccccc} s^n : & 1 & a_2 & a_4 & \dots & & \\ s^{n-1} : & a_1 & a_3 & a_5 & \dots & & \end{array}$$

Routh array: method (cont'd)

- Then add subsequent rows to complete the Routh array:

$$\begin{array}{l}
 s^n : \quad 1 \quad a_2 \quad a_4 \quad \dots \\
 s^{n-1} : \quad a_1 \quad a_3 \quad a_5 \quad \dots \\
 s^{n-2} : \quad b_1 \quad b_2 \quad b_3 \quad \dots \\
 s^{n-3} : \quad c_1 \quad c_2 \quad c_3 \quad \dots \\
 \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \\
 s^2 : \quad * \quad * \\
 s^1 : \quad * \\
 s^0 : \quad *
 \end{array}$$

Routh array: method (cont'd)

- Compute elements for the 3rd row:

$$b_1 = -\frac{1 \times a_3 - a_2 a_1}{a_1},$$

$$b_2 = -\frac{1 \times a_5 - a_4 a_1}{a_1},$$

$$b_3 = -\frac{1 \times a_7 - a_6 a_1}{a_1}$$

...

$$\begin{array}{l} S^n : \quad 1 \quad a_2 \quad a_4 \quad \dots \\ S^{n-1} : \quad a_1 \quad a_3 \quad a_5 \quad \dots \\ S^{n-2} : \quad b_1 \quad b_2 \quad b_3 \quad \dots \\ S^{n-3} : \quad c_1 \quad c_2 \quad c_3 \quad \dots \\ \vdots \\ S^2 : \quad * \quad * \\ S^1 : \quad * \\ S^0 : \quad * \end{array}$$

Routh array: method (cont'd)

- Compute elements for the 4th row:

$$c_1 = -\frac{a_1 \times b_2 - a_3 b_1}{b_1},$$

$$c_2 = -\frac{a_1 \times b_3 - a_5 b_1}{b_1},$$

$$c_3 = -\frac{a_1 \times b_4 - a_7 b_1}{b_1}$$

...

$$\begin{array}{rcl}
 s^n & : & 1 \quad a_2 \quad a_4 \quad \dots \\
 s^{n-1} & : & a_1 \quad a_3 \quad a_5 \quad \dots \\
 s^{n-2} & : & b_1 \quad b_2 \quad b_3 \quad \dots \\
 s^{n-3} & : & c_1 \quad c_2 \quad c_3 \quad \dots \\
 & : & \vdots \quad \vdots \quad \vdots \\
 s^2 & : & * \quad * \\
 s^1 & : & * \\
 s^0 & : & *
 \end{array}$$

Method for determining the Routh array

$$P(s) = a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0$$

s^n	a_n	a_{n-2}	a_{n-4}	$\dots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	$\dots$
s^{n-2}	b_1	b_2	b_3	$\dots$
s^{n-3}	c_1	c_2	c_3	$\dots$
$\vdots$				
s^0				

$$b_1 = \frac{a_{n-2}a_{n-1} - a_n a_{n-3}}{a_{n-1}}$$

$$b_2 = \frac{a_{n-4}a_{n-1} - a_n a_{n-5}}{a_{n-1}}$$

$$c_1 = \frac{a_{n-3}b_1 - a_{n-1}b_2}{b_1}$$

The number of RHP roots of $P(s)$ is the number of algebra sign changes in the elements of the left column of the array.

Method for determining the Routh array

Example 1:

Given the characteristic equation,

$$a(s) = s^6 + 4s^5 + 3s^4 - 2s^3 + s^2 + 4s + 4$$

is the system described by this characteristic equation stable?

Answer:

- One coefficient (-2) is negative.
- Therefore, the system **does not** satisfy the necessary condition for stability.

Method for determining the Routh array

Example 2:

Given the characteristic equation,

$$a(s) = s^6 + 4s^5 + 3s^4 + 2s^3 + s^2 + 4s + 4$$

is the system described by this characteristic equation stable?

Answer:

- All the coefficients are positive and nonzero.
- Therefore, the system **satisfies** the necessary condition for stability.
- We should determine whether any of the coefficients of the first column of the Routh array are negative.

Method for determining the Routh array

Example 2 (cont'd): Routh array

$$a(s) = s^6 + 4s^5 + 3s^4 + 2s^3 + s^2 + 4s + 4$$

$$\begin{array}{l} s^6 : \quad 1 \quad 3 \quad 1 \quad 4 \\ s^5 : \quad 4 \quad 2 \quad 4 \quad 0 \\ s^4 : \quad ? \quad ? \quad ? \\ s^3 : \quad ? \quad ? \quad ? \\ s^2 : \quad ? \quad ? \\ s^1 : \quad ? \quad ? \\ s^0 : \quad ? \end{array}$$

Example 2 (cont'd): Routh array

$$a(s) = s^6 + 4s^5 + 3s^4 + 2s^3 + s^2 + 4s + 4$$

$$s^6 : \quad 1 \quad 3 \quad 1 \quad 4$$

$$s^5 : \quad 4 \quad 2 \quad 4 \quad 0$$

$$s^4 : \quad 5/2 \quad 0 \quad 4$$

$$s^3 : \quad ? \quad ? \quad ?$$

$$s^2 : \quad ? \quad ?$$

$$s^1 : \quad ? \quad ?$$

$$s^0 : \quad ?$$

$$-\frac{1 \times 2 - 3 \times 4}{4} = -\frac{-10}{4} = \frac{5}{2}$$

$$-\frac{1 \times 4 - 1 \times 4}{4} = -\frac{0}{4} = 0$$

$$-\frac{1 \times 0 - 4 \times 4}{4} = -\frac{-16}{4} = 4$$

Method for determining the Routh array

Example 2 (cont'd): Routh array

$$a(s) = s^6 + 4s^5 + 3s^4 + 2s^3 + s^2 + 4s + 4$$

$$\begin{array}{l} s^6 : \quad 1 \quad 3 \quad 1 \quad 4 \\ s^5 : \quad 4 \quad 2 \quad 4 \quad 0 \\ s^4 : \quad 5/2 \quad 0 \quad 4 \\ s^3 : \quad 2 \quad -12/5 \quad 0 \\ s^2 : \quad ? \quad ? \\ s^1 : \quad ? \quad ? \\ s^0 : \quad ? \end{array}$$

$$-\frac{4 \times 0 - 2 \times 5/2}{5/2} = -(-2) = 2$$

$$-\frac{4 \times 4 - 4 \times 5/2}{5/2} = -\frac{32 - 20}{5} = \frac{-12}{5}$$

Method for determining the Routh array

Example 2 (cont'd): Routh array

$$a(s) = s^6 + 4s^5 + 3s^4 + 2s^3 + s^2 + 4s + 4$$

$$s^6 : \quad 1 \quad 3 \quad 1 \quad 4$$

$$s^5 : \quad 4 \quad 2 \quad 4 \quad 0$$

$$s^4 : \quad 5/2 \quad 0 \quad 4$$

$$s^3 : \quad 2 \quad -12/5 \quad 0$$

$$s^2 : \quad 3 \quad 4$$

$$s^1 : \quad -76/15 \quad 0$$

$$s^0 : \quad 4$$

The elements of
the 1st column
are not all positive:

the system is unstable

Method for determining the Routh array

Example:

$$P(s) = 2s^4 + 3s^3 + 5s^2 + 2s + 6$$

s^4	2	5	6
s^3	3	2	
s^2	$\frac{15 - 4}{3} = \frac{11}{3}$	$\frac{18 - 0}{3} = 6$	
s^1	$\frac{-32}{11}$		
s^0	6		

Two roots in the RHP

Method for determining the Routh array

Example:

$$P(s) = s^4 + 2s^3 + 3s^2 + 4s + 1$$

$$s^4 \quad 1 \quad 3 \quad 1$$

$$s^3 \quad 2 \quad 4$$

$$s^2 \quad 1 \quad 1$$

$$s^1 \quad 2$$

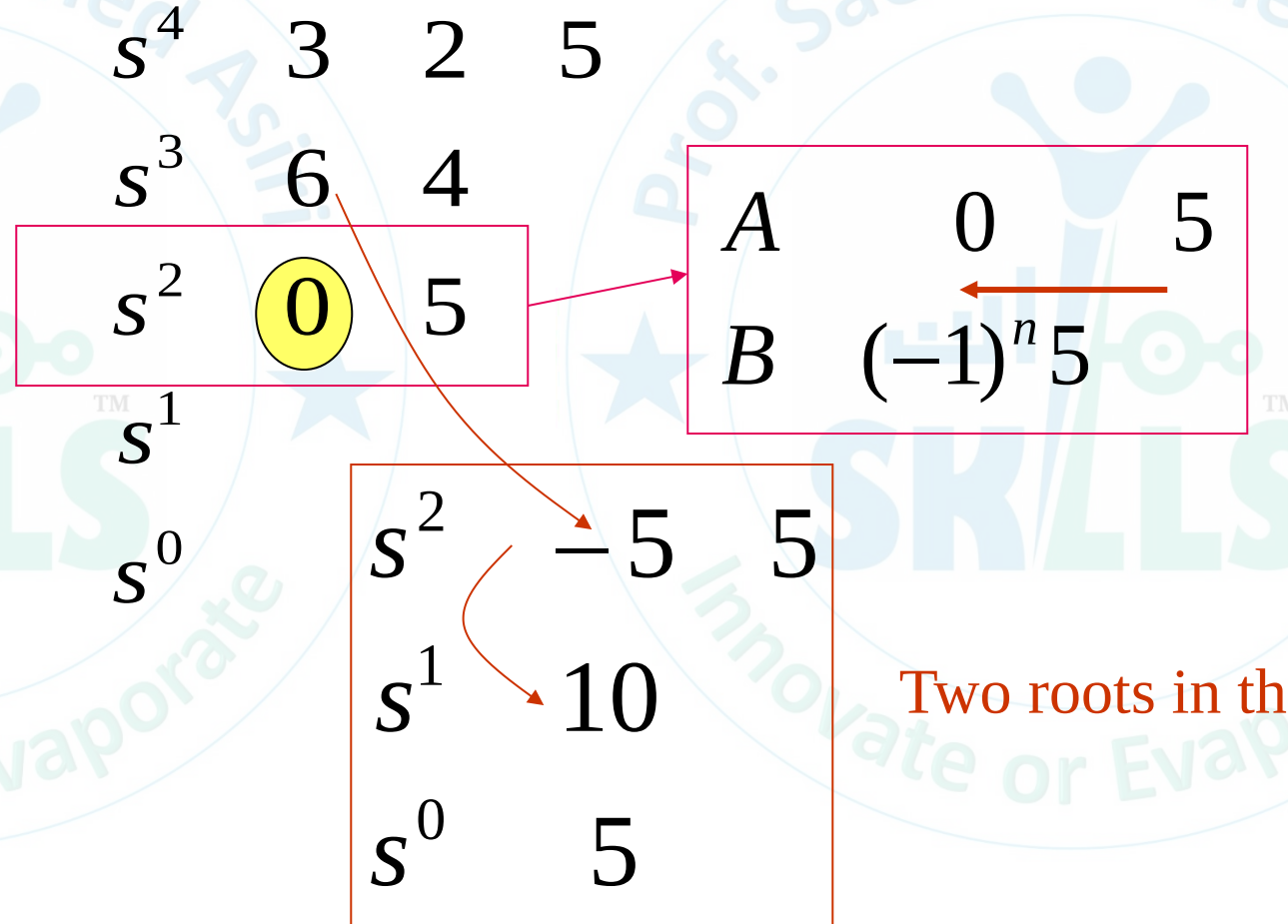
$$s^0 \quad 1$$

no root in the RHP

Method for determining the Routh array

Example:

$$P(s) = 3s^4 + 6s^3 + 2s^2 + 4s + 5$$



Method for determining the Routh array

Example:

$$P(s) = s^5 + 2s^4 + 8s^3 + 11s^2 + 16s + 12$$

$$s^5 \quad 1 \quad 8 \quad 16$$

$$s^4 \quad 2 \quad 11 \quad 12$$

$$s^3 \quad 2.5 \quad 10$$

$$s^2 \quad 3 \quad 12$$

$$s^1 \quad 0 \quad 0$$

$$s^0$$

factor

$$3s^2 + 12$$

Method for determining the Routh array

Example: $P(s) = s^5 + 2s^4 + 8s^3 + 11s^2 + 16s + 12$

s^5	1	8	16
s^4	2	11	12
s^3	2.5	10	
s^2	3	12	
s^1	6		
s^0	12		

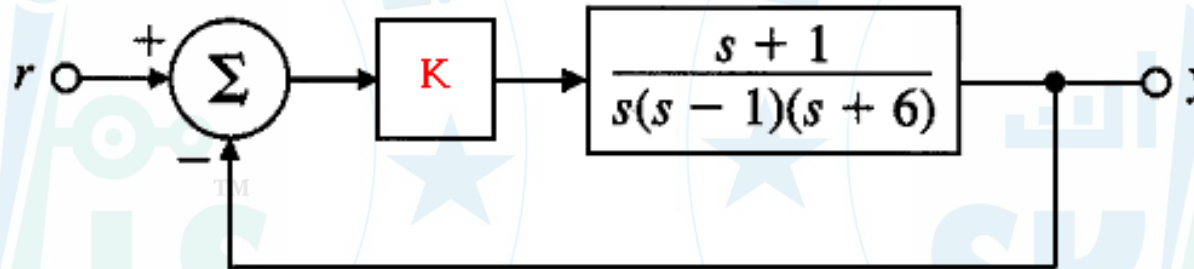
$$\frac{d}{ds}(3s^2 + 12) = 6s$$

no root in the RHP

Stability versus Parameter Range

Example:

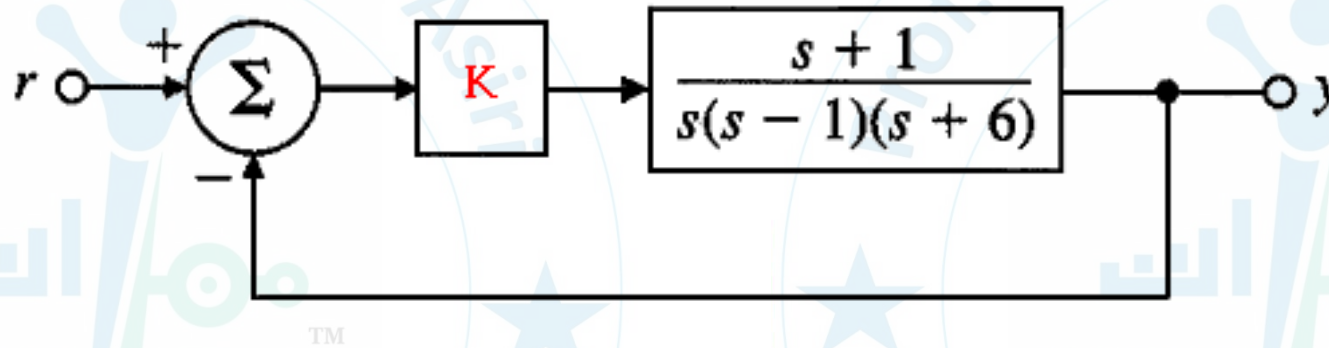
Consider a feedback system such as:



The stability properties of this system are a function of the proportional feedback gain K . Determine the range of K over which the system is stable.

Stability versus Parameter Range

Example:



- The characteristic equation for the system is given by:

$$1 + K \frac{s+1}{s(s-1)(s+6)} = 0$$

Stability versus Parameter Range

Example:

$$1 + K \frac{s+1}{s(s-1)(s+6)} = 0$$

- Expressing the characteristic equation in polynomial form, we obtain:

$$s^3 + 5s^2 + (K-6)s + K = 0$$

Stability versus Parameter Range

Example:

$$a(s) = s^3 + 5s^2 + (K - 6)s + K$$

Routh array

$$\begin{array}{l} s^3 : \quad 1 \quad K - 6 \\ s^2 : \quad 5 \quad K \\ s^1 : \quad (4K - 30)/5 \\ s^0 : \quad K \end{array}$$

Therefore, the **system is stable if and only if**

$$\frac{4K - 30}{5} > 0 \text{ and } K > 0$$

$$\Rightarrow K > 7.5 \text{ and } K > 0$$

$$\Rightarrow K > 7.5$$

Stability versus Parameter Range

Example:

$$a(s) = s^3 + 5s^2 + (K - 6)s + K$$

- Solving for the roots using MATLAB gives:

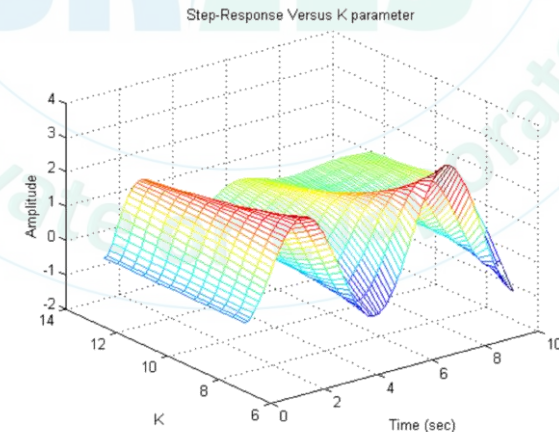
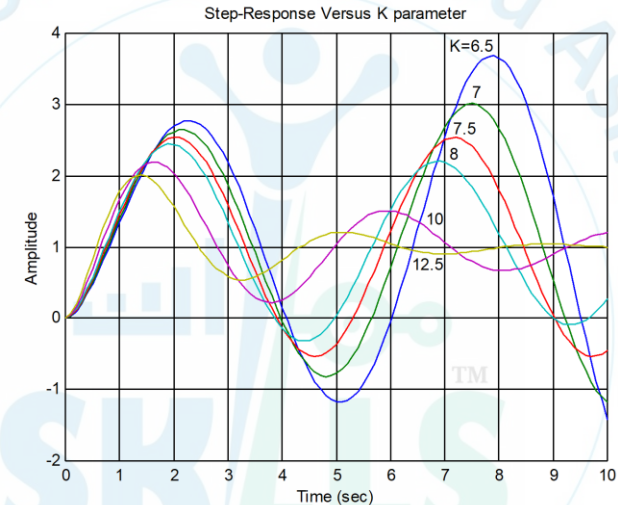
-5 and $\pm 1.22j$ for $K=7.5$

=> The system is unstable (or critically stable) for $K=7.5$

-4.06 and $-0.47 \pm 1.7j$ for $K=13$

-1.90 and $-1.54 \pm 3.27j$ for $K=25$

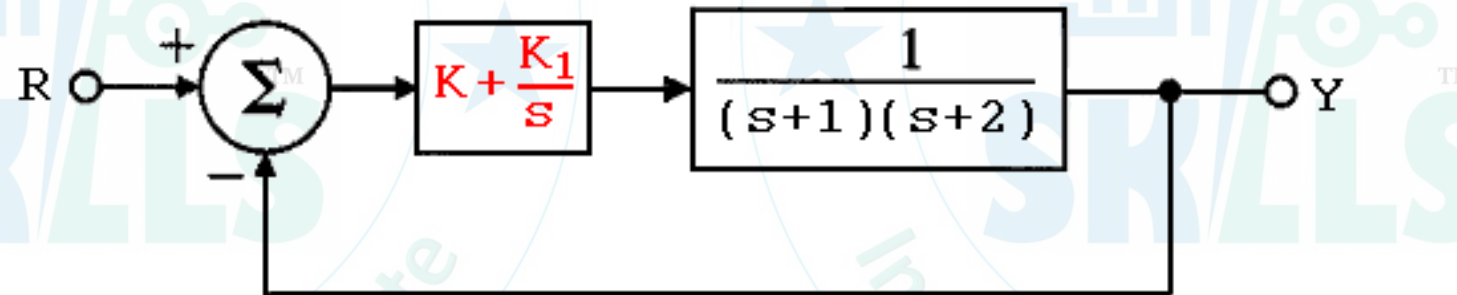
=> The system is stable for both $K=13$ and $K=25$



Stability versus Parameter Range

Example:

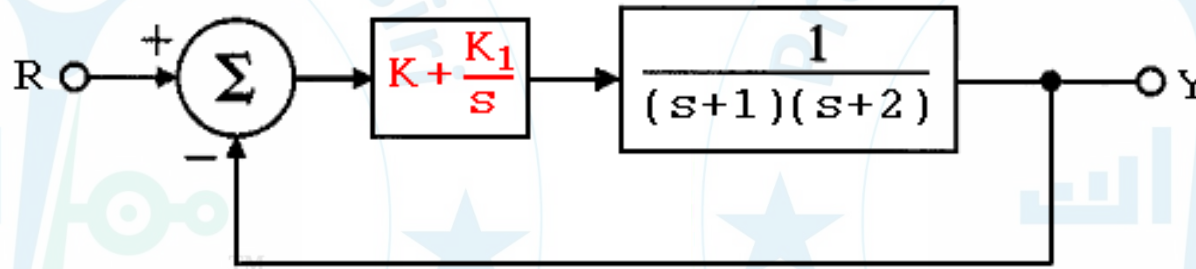
Consider a Proportional-Integral (PI) control such as:



Find the range of the controller gains (K, K_1) so that the PI feedback system is stable.

Stability versus Parameter Range

Example:



The characteristic equation for the system is given by:

$$1 + \left(K + \frac{K_1}{s} \right) \frac{1}{(s+1)(s+2)} = 0$$

Stability versus Parameter Range

Example:

$$1 + \left(K + \frac{K_1}{s} \right) \frac{1}{(s+1)(s+2)} = 0$$

Expressing the characteristic equation in polynomial form, we obtain:

$$s^3 + 3s^2 + (2 + K)s + K_1 = 0$$

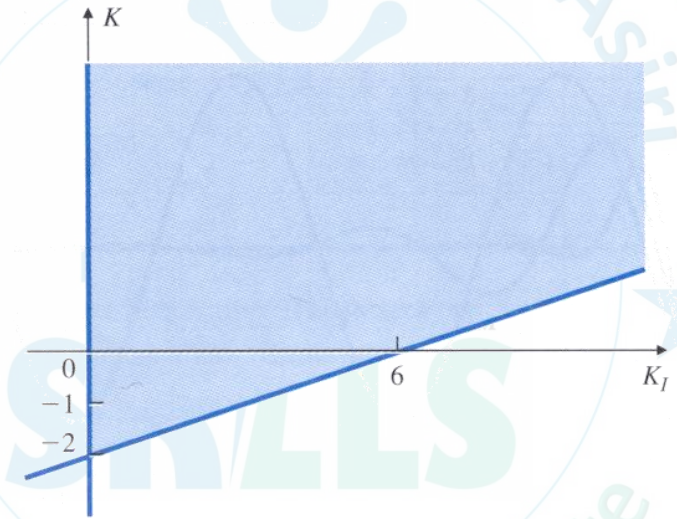
Stability versus Parameter Range

Example:

$$a(s) = s^3 + 5s^2 + (K - 6)s + K$$

The corresponding Routh array is:

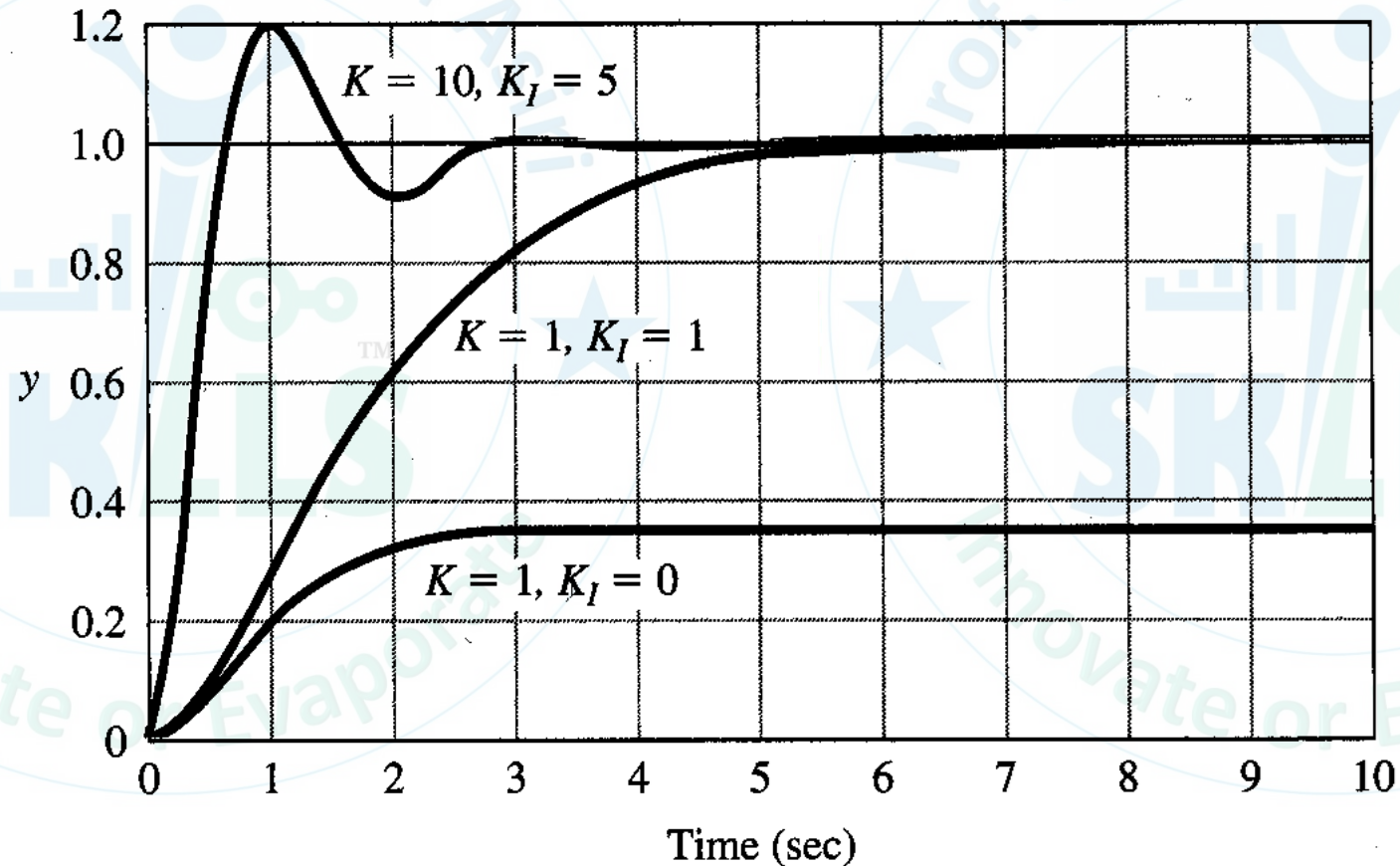
$s^3 :$	1	$2 + K$
$s^2 :$	3	K_1
$s^1 :$	$(6 + 3K - K_1)/3$	
$s^0 :$	K_1	



For **stability**, we must have:

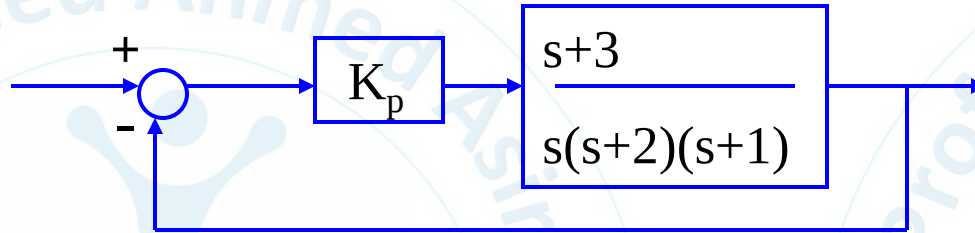
$$K_1 > 0 \text{ and } K > \frac{K_1}{3} - 2$$

Example 6: Transient Response for the System



Stability versus Parameter Range

Example:



Q : find range of K_p for stability

$$\text{Sol: } G_{c.l.}(s) = \frac{K_p(s+3)}{s(s+2)(s+1) + K_p(s+3)}$$

$$\begin{aligned} \therefore \text{char.poly } d(s) &= s(s+2)(s+1) + K_p(s+3) \\ &= s^3 + 3s^2 + (2 + K_p)s + 3K_p \end{aligned}$$

Stability versus Parameter Range

Example:

Routh table :

$$\begin{array}{l} s^3 : \quad 1 \\ s^2 : \quad 3 \\ s^1 : \quad \frac{3(2 + K_p) - 3K_p}{3} \\ s^0 : \quad 3K_p \end{array}$$

$$\begin{array}{l} 2 + K_p \\ 3K_p \end{array}$$

For A.S.: 1st col. > 0

$$\therefore \begin{cases} 3K_p > 0 \\ 3(2 + K_p) - 3K_p > 0 \end{cases}$$

$$\Rightarrow K_p > 0$$

Stability versus Parameter Range

Example:

e.g. $d(s) = s^3 + 3ks^2 + (k+2)s + 4$

remember 3rd order Routh criteria?

1) all coeff > 0

2) prod of mid two $>$ outer prod.

$\therefore$ For A.S.: we need:

$$1) \quad 3k > 0 \quad \Rightarrow k > 0$$

$$k + 2 > 0 \quad \Rightarrow k > -2$$

$$2) \quad 3k(k+2) > 1 \times 4$$

$$3k^2 + 6k - 4 > 0$$

$$k^2 + 2k - \frac{4}{3} > 0$$

$$(k+1)^2 - \frac{7}{3} > 0$$

Stability versus Parameter Range

Example:

$$(k+1)^2 > \frac{7}{3}$$

$$k+1 > \sqrt{\frac{7}{3}} \Rightarrow k > \sqrt{\frac{7}{3}} - 1$$

$$\text{or } k+1 < -\sqrt{\frac{7}{3}} \Rightarrow k < -\sqrt{\frac{7}{3}} - 1$$

but $k > 0$ and $k > -2$ also.

over all we need

$$k > \sqrt{\frac{7}{3}} - 1 \approx 0.528$$

$$k > 0.5$$

for A.S.

Stability versus Parameter Range

Example:

If we set $3k(k + 2) = 4$

we get: $k = \sqrt{\frac{7}{3}} - 1$

At this k , $d(s)$ is M.S.

And s^1 row = 0

From s^2 row: $3ks^2 + 4 = A(s) = 0$

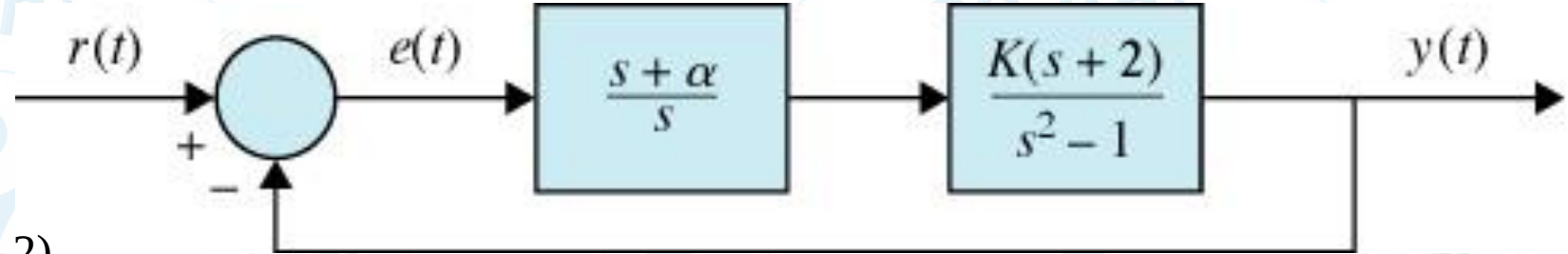
$$\Rightarrow s = \pm j\sqrt{\frac{4}{3k}}$$

leads to sustained oscillation

$$\text{osci freq: } \omega = \sqrt{\frac{4}{3k}} = \sqrt{\frac{4}{3(\sqrt{\frac{7}{3}} - 1)}}$$

Stability versus Parameter Range

Example:



$$H(s) = \frac{(s + \alpha)K(s + 2)}{s(s^2 - 1) + (s + \alpha)K(s + 2)}$$

$$\begin{aligned} d(s) &= s(s^2 - 1) + (s + \alpha)K(s + 2) \\ &= s^3 + Ks^2 + (K(\alpha + 2) - 1)s + 2\alpha K \end{aligned}$$

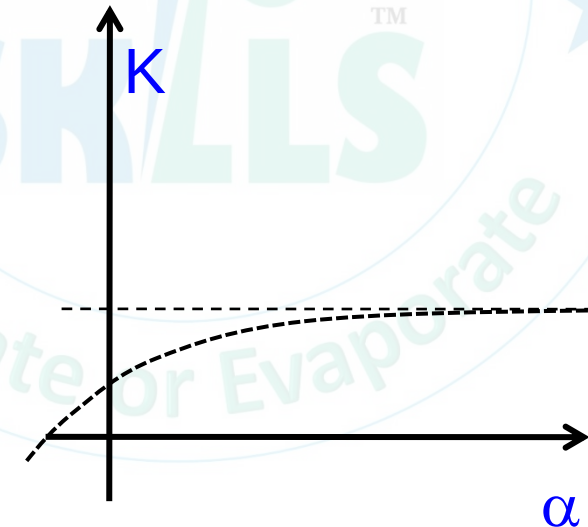
Routh Criteria :

$$K > 0$$

$$K(\alpha + 2) - 1 > 0 \Rightarrow K > \frac{1}{\alpha + 2}$$

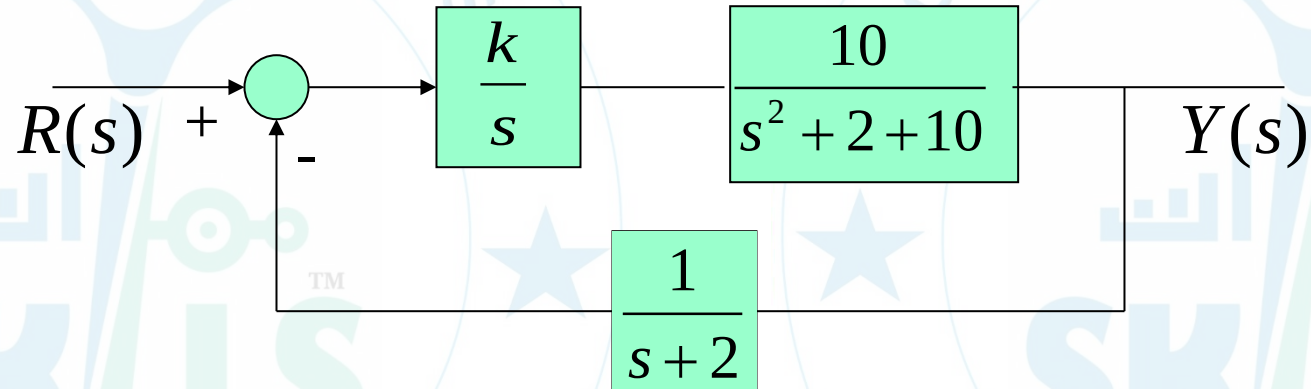
$$2\alpha K > 0 \Rightarrow \alpha > 0$$

$$K(K(\alpha + 2) - 1) > 2\alpha K \Rightarrow K > \frac{1 + 2\alpha}{\alpha + 2}$$



Stability versus Parameter Range

Example:



$$\frac{Y(s)}{R(s)} = \frac{10k}{s(s^2 + s + 10)(s + 2) + 10k}$$

END