

State Space Analysis

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State Space Representation

Obtaining State Space Representation

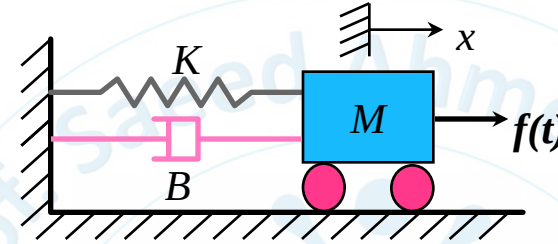
- Identify State Variables
 - Rule of Thumb:
 - Nth order ODE requires N state variables.
 - Position and velocity of inertia elements are natural state variables for translational mechanical systems.
- Eliminate all algebraic equations written in the modeling process.
- Express the resulting differential equations in terms of state variables and inputs in coupled first order ODEs.
- Express the output variables as algebraic functions of the state variables and inputs.
- For linear systems, put the equations in matrix form.

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A} \cdot \underbrace{\mathbf{x}}_{\text{State Vector}} + \mathbf{B} \cdot \underbrace{\mathbf{u}}_{\text{Input Vector}} \\ \underbrace{\mathbf{y}}_{\text{Output Vector}} &= \mathbf{C} \cdot \mathbf{x} + \mathbf{D} \cdot \mathbf{u}\end{aligned}$$

State Space Representation

Mechanical system Example:

$$\text{EOM: } M\ddot{x} + B\dot{x} + Kx = f(t)$$



Q: What information about the mass do we need to know to be able to solve for $x(t)$ for $t \geq t_0$?

Input: $f(t)$, $t \geq t_0$

Initial Conditions (ICs): $x(t_0)$ $q_1 = x(t)$
 $\dot{x}(t_0)$ $q_2 = \dot{x}(t)$

Rule of Thumb

Number of state variables = Sum of orders of EOMs

State Space Representation

Mechanical system Example:

EOM $M \ddot{x} + B \dot{x} + K x = f(t)$

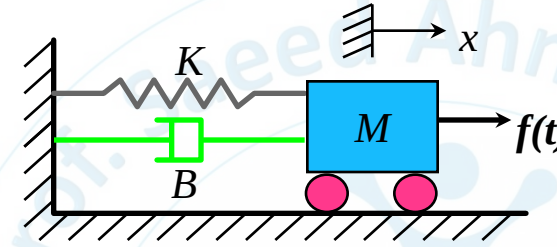
State Variables: $q_1 = x,$
 $q_2 = \dot{x}$

Outputs: $y_1 = x, \quad y_2 = -B \dot{x}$

State Space Representation:

$$\begin{cases} \dot{q}_1 = \dot{x} = q_2 \\ \dot{q}_2 = \ddot{x} = \frac{1}{M} (-B q_2 - K q_1) + \frac{1}{M} f(t) \end{cases} \quad \text{State equation}$$

$$\begin{cases} y_1 = x = q_1 \\ y_2 = -B \dot{x} = -B q_2 \end{cases} \quad \text{Output equation}$$



Matrix Form

$$\underbrace{\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{K}{M} & -\frac{B}{M} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix}}_{\mathbf{B}} \underbrace{[f]}_{\mathbf{u}}$$

$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -B \end{bmatrix}}_{\mathbf{C}} \underbrace{\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}}_{\mathbf{x}} + \underbrace{0}_{\mathbf{D}} \cdot \mathbf{u}$$

State Space Representation

Exercise

Represent the 2 DOF suspension system in a state space representation.

Let the system output be the relative position of mass M_1 with respect to M_2 .

$$\begin{aligned} M_1 \ddot{x}_1 + B_1 \dot{x}_1 - B_1 \dot{x}_2 + K_1 x_1 - K_1 x_2 &= 0 \\ M_2 \ddot{x}_2 - B_1 \dot{x}_1 + B_1 \dot{x}_2 - K_1 x_1 + (K_1 + K_2) x_2 &= K_2 x_p \end{aligned}$$

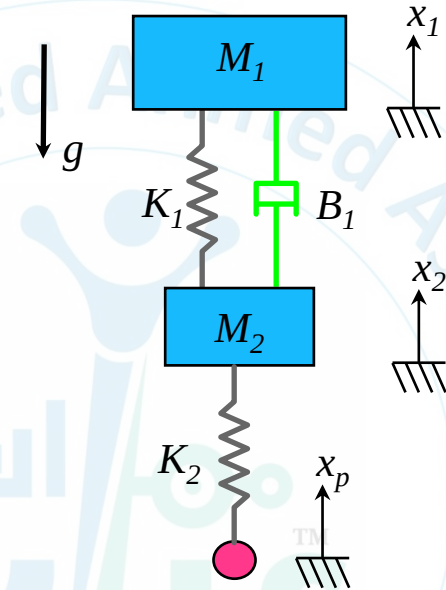
$$q_1 = x_1, \quad q_2 = \dot{x}_1, \quad q_3 = x_2, \quad q_4 = \dot{x}_2$$

State Variables:

Output: $y = x_1 - x_2$ Input: x_p

State Space Representation:

$$\begin{aligned} \underbrace{\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix}}_{\dot{\mathbf{x}}} &= \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{K_1}{M_1} & -\frac{B_1}{M_1} & \frac{K_1}{M_1} & \frac{B_1}{M_1} \\ 0 & 0 & 0 & 1 \\ \frac{K_1}{M_2} & \frac{B_1}{M_2} & -\frac{K_1 + K_2}{M_2} & -\frac{B_1}{M_2} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{K_2}{M_2} \end{bmatrix}}_{\mathbf{B}} \underbrace{x_p}_{\mathbf{u}} \\ y &= \underbrace{\begin{bmatrix} 1 & 0 & -1 & 0 \end{bmatrix}}_{\mathbf{C}} \cdot \mathbf{x} + \underbrace{0}_{\mathbf{D}} \cdot u \end{aligned}$$



Input/Output Representation

Input/Output Model

Uses one n th order ODE to represent the relationship between the input variable, $u(t)$, and the output variable, $y(t)$, of a system.

For linear time-invariant (LTI) systems, it can be represented by :

where

$$a_n y^{(n)} + \dots + a_2 \ddot{y} + a_1 \dot{y} + a_0 y = b_m u^{(m)} + \dots + b_2 \ddot{u} + b_1 \dot{u} + b_0 u(t)$$

- To solve an input/output differential equation, we need to know

Input:

$$u(t), \quad t \geq 0$$

Initial Conditions (ICs):

$$\underbrace{y(0), \dot{y}(0), \dots, y^{(n-1)}(0)}_n$$

- To obtain I/O models:
 - Identify input/output variables.
 - Derive equations of motion.
 - Combine equations of motion by eliminating all variables except the input and output variables and their derivatives.

Input/Output Representation

Example

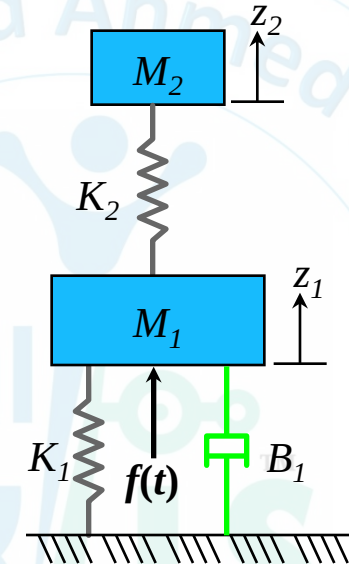
Vibration Absorber

EOM:

$$\begin{aligned} M_1 \ddot{z}_1 + B_1 \dot{z}_1 + (K_1 + K_2)z_1 - K_2 z_2 &= f(t) \\ M_2 \ddot{z}_2 + K_2 z_2 - K_2 z_1 &= 0 \end{aligned}$$

- Find input/output representation between input $f(t)$ and output z_2 .
 - Need to eliminate z_1 and its time derivative of all orders

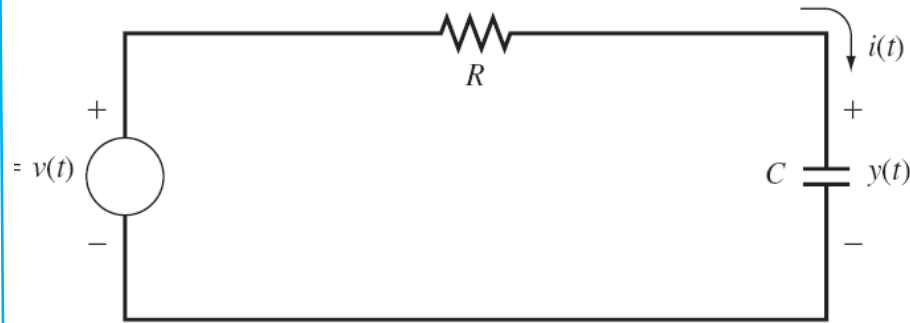
$$\begin{aligned} z_1 &= \frac{1}{K_2} \{M_2 \ddot{z}_2 + K_2 z_2\} \\ \dot{z}_1 &= \frac{1}{K_2} \{M_2 \dot{z}_2^{(3)} + K_2 \dot{z}_2\}, \\ \ddot{z}_1 &= \frac{1}{K_2} \{M_2 \ddot{z}_2^{(4)} + K_2 \ddot{z}_2\} \end{aligned}$$



$$z_2^{(4)} + \frac{B_1}{M_1} z_2^{(3)} + \frac{M_1 K_2 + M_2 (K_1 + K_2)}{M_1 M_2} \ddot{z}_2 + \frac{B_1 K_2}{M_1 M_2} \dot{z}_2 + \frac{K_1 K_2}{M_1 M_2} z_2 = \frac{K_2}{M_1 M_2} f$$

State Space Representation

Electrical system Example:



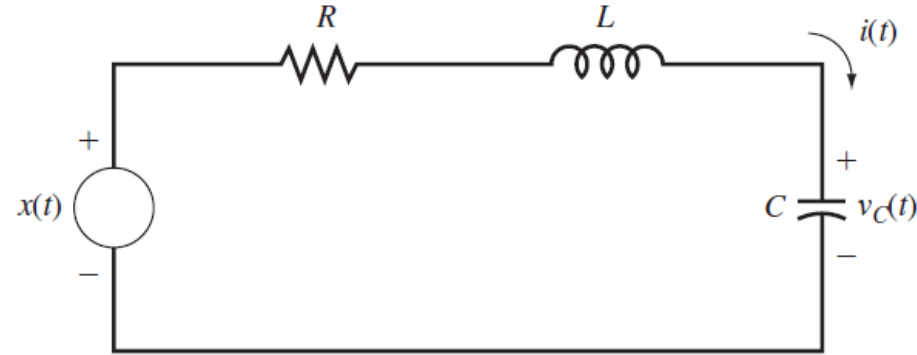
$$\frac{dy(t)}{dt} + \frac{1}{RC} y(t) = \frac{1}{RC} v(t)$$

$$\mathbf{A} = [-a_0] = \left[-\frac{1}{RC} \right]$$

$$\mathbf{B} = [b_0] = \left[\frac{1}{RC} \right] \quad \mathbf{C} = [1] \quad \mathbf{D} = 0$$

$$\begin{bmatrix} \dot{x}_1(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{RC} \end{bmatrix} \begin{bmatrix} x_1(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{RC} \end{bmatrix} v(t)$$

$$y(t) = [1] \begin{bmatrix} x_1(t) \end{bmatrix}$$



$$\frac{d^2 y(t)}{dt^2} + \left(\frac{R}{L} \right) \frac{dy(t)}{dt} + \frac{1}{LC} y(t) = \frac{1}{LC} v(t)$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1/LC & -R/L \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 0 \\ b_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/LC \end{bmatrix} \quad \mathbf{C} = [1 \quad 0] \quad \mathbf{D} = 0$$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1/LC & -R/L \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1/LC \end{bmatrix} v(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Input/Output Models vs State-Space Models

- **State Space Models:**

- consider the internal behavior of a system
- can easily incorporate complicated output variables
- have significant computation advantage for computer simulation
- can represent multi-input multi-output (MIMO) systems and nonlinear systems

- **Input/Output Models:**

- are conceptually simple
- are easily converted to frequency domain transfer functions that are more intuitive to practicing engineers
- are difficult to solve in the time domain (solution: Laplace transformation)

Conversion between different models

Converting From	Converting to	Matlab function
Transfer Function	Zero-pole-gain	$[z, p, k] = \text{tf2zp}(\text{num}, \text{den})$
Transfer Function	State Space	$[A, B, C, D] = \text{tf2ss}(\text{num}, \text{den})$
Zero-pole-gain	Transfer Function	$[\text{num}, \text{den}] = \text{zp2tf}(z, p, k)$
Zero-pole-gain	State Space	$[A, B, C, D] = \text{zp2ss}(z, p, k)$
State Space	Transfer Function	$[\text{num}, \text{den}] = \text{ss2tf}(A, B, C, D)$
State Space	Zero-pole-gain	$[z, p, k] = \text{ss2zp}(A, B, C, D)$

State Space Equations

Dynamic Equations

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

Observation Equations

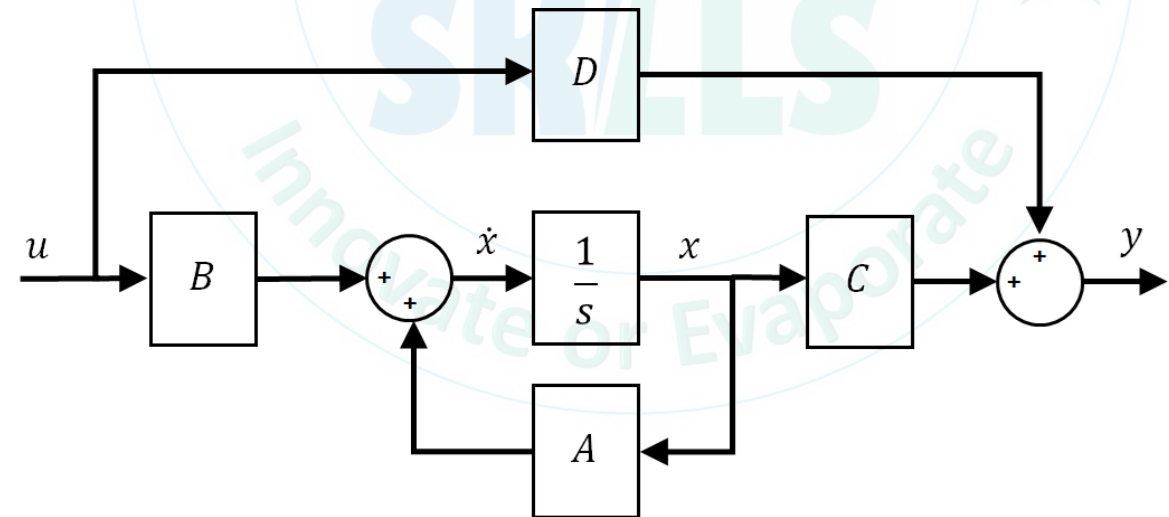
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

$\mathbf{x}$ is the state vector of the system, $n \times 1$;
 n is the order of the system;
 $\mathbf{u}$ is the input vector, $m \times 1$;
 $\mathbf{y}$ is the output vector, $p \times 1$;
 $\mathbf{A}$ is the system (or coefficient) matrix, $n \times n$;
 $\mathbf{B}$ is the input (or driving) matrix, $n \times m$;
 $\mathbf{C}$ is the observation matrix, $p \times n$;
 $\mathbf{D}$ is the feedforward matrix, $p \times m$.

MATLAB function:

```

SYS = SS(A,B,C,D
SYS = SS(A,B,C,D,T)
SYS = SS
SYS = SS(D)
SYS =
SS(A,B,C,D,LTISYS)
SYS = SS(SYS)
  
```



State Space Model to Transfer Function Model

$$s\mathbf{x}(s) - \mathbf{x}(0) = A \mathbf{x}(s) + B \mathbf{u}(s)$$

Solving for $\mathbf{x}(s)$,

$$\mathbf{x}(s) = \text{inv}(sI-A) \mathbf{x}(0) + \text{inv}(sI-A) B \mathbf{u}(s).$$

So,

$$\begin{aligned} \mathbf{y}(s) &= C \mathbf{x}(s) + D \mathbf{u}(s) \\ &= C \text{inv}(sI-A) B \mathbf{u}(s) + D \mathbf{u}(s) \text{ for } \mathbf{x}(0) = \mathbf{0}. \\ &= [C \text{inv}(sI-A) B + D] \mathbf{u}(s) \\ &= \{ [C \text{adj}(sI-A) B + D \det(sI-A)] / \det(sI-A) \} \mathbf{u}(s) \end{aligned}$$

$\det(sI-A) = 0$ is the *characteristic equation* of the system and
 $[C \text{adj}(sI-A) B + D \det(sI-A)]$ is the *pxm matrix of system zeros*

MATLAB function:

```
[NUM,DEN] =  
SS2TF(A,B,C,D,iu)
```

State Space Model to Transfer Function Model

State equations as functions of time

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

In the s-domain

$$sX - X_0 = AX + BU$$

$$X(sI - A) = BU + X_0$$

$$X = (sI - A)^{-1}BU + (sI - A)^{-1}X_0$$

$$Y = CX + DU$$

$$Y = C((sI - A)^{-1}BU + (sI - A)^{-1}X_0) + DU$$

$$Y = (C(sI - A)^{-1}B + D)U + C(sI - A)^{-1}X_0$$

Assuming the system starts at rest,

$$Y = (C(sI - A)^{-1}B + D)U$$

$$\frac{Y}{U} = (C(sI - A)^{-1}B + D) \quad (\text{the transfer function})$$



State Space Modeling

Example:

State-space representations are, in general, not unique. One system can be represented in several possible ways. For example, consider the following systems:

a. $\dot{x} = -5x + 3u$
 $y = 7x$

b. $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -5 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} u$
 $y = [7 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

c. $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -5 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} u$
 $y = [7 \quad 3] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Show that these systems will result in the same transfer function. We will explore this phenomenon

a.

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} = 7(s+5)^{-1}3 = \frac{21}{(s+5)}$$

Pole at $s=-5$

1 state variable & one pole

b.

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} = [7 \quad 0] \begin{bmatrix} s+5 & 0 \\ 0 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = [7 \quad 0] \begin{bmatrix} \frac{1}{s+5} & 0 \\ 0 & \frac{1}{s+1} \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{7}{s+5} & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \frac{21}{s+5}$$

c) $G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} = [7 \quad 3] \begin{bmatrix} s+5 & 0 \\ 0 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 0 \end{bmatrix} = [7 \quad 3] \begin{bmatrix} \frac{1}{s+5} & 0 \\ 0 & \frac{1}{s+1} \end{bmatrix} \begin{bmatrix} 3 \\ 0 \end{bmatrix}$

$$= [7 \quad 0] \begin{bmatrix} \frac{3}{s+5} \\ 0 \end{bmatrix} = \frac{21}{s+5}$$

2 state variables & one pole

but, two eigenvalues



>> A=[-5 0;0 -1];

>> eig(A)

ans = -5 -1

Controllability and Observability

The Controllability Matrix

An n th-order plant whose state equation is

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

is completely controllable if the matrix

$$\mathbf{C}_M = [\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B} \quad \dots \quad \mathbf{A}^{n-1}\mathbf{B}]$$

is of rank n , where $\mathbf{C}_M$ is called the *controllability* matrix.

Controllability and Observability

Example:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u$$

$$\mathbf{C}_M = [\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B}] = \begin{bmatrix} 0 & 1 & -2 \\ 1 & -1 & 1 \\ 1 & -2 & 4 \end{bmatrix}$$

The rank of $\mathbf{C}_M$ equals the number of linearly independent rows or columns. The rank can be found by finding the highest-order square submatrix that is nonsingular.

The determinant of $\mathbf{C}_M = -1$. Since the determinant is not zero, the 3X3 matrix is nonsingular, and the rank of $\mathbf{C}_M$ is 3.

A=[-1 1 0;0 -1 0;0 0 -2]

B=[0;1;1]

Cm=ctrb(A,B)

Rank=rank(Cm)

% Define compensated A matrix.

% Define compensated B matrix.

% Calculate controllability matrix.

% Find rank of controllability matrix.

Controllability and Observability

The Observability Matrix

An n th-order plant whose state and output equations are, respectively,

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x}$$

is completely observable⁶ if the matrix

$$\mathbf{O}_M = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix}$$

is of rank n , where $\mathbf{O}_M$ is called the *observability matrix*.

Controllability and Observability

Example: The state and output equations for the system are

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u = \begin{bmatrix} 0 & 1 \\ -5 & -21/4 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \mathbf{C}\mathbf{x} = \begin{bmatrix} 5 & 4 \end{bmatrix} \mathbf{x}$$

The observability matrix, $\mathbf{O}_M$, for this system is

$$\mathbf{O}_M = \begin{bmatrix} \mathbf{C} \\ \mathbf{C}\mathbf{A} \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ -20 & -16 \end{bmatrix}$$

The rank of $\mathbf{O}_M$ equals the number of linearly independent rows or columns. The rank can be found by finding the highest-order square submatrix that is nonsingular. The determinant of $\mathbf{O}_M = 0$. Since the determinant is zero, the 2 X 2 matrix is singular, and the rank of $\mathbf{C}_M$ is equal to 1.

A=[0 1;-5 -21/4]

C=[5 4]

Om=obsv(A,C)

Rank=rank(Om)

% Define compensated A matrix.

% Define compensated C matrix.

% Calculate observability matrix.

% Find rank of observability matrix.

Controllability and Observability

Example:

The state and output equations for the system are

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -3 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \mathbf{C}\mathbf{x} = [0 \ 5 \ 1]\mathbf{x}$$

$$\mathbf{O}_M = \begin{bmatrix} \mathbf{C} \\ \mathbf{C}\mathbf{A} \\ \mathbf{C}\mathbf{A}^2 \end{bmatrix} = \begin{bmatrix} 0 & 5 & 1 \\ -4 & -3 & 3 \\ -12 & -13 & -9 \end{bmatrix}$$

The rank of $\mathbf{O}_M$ equals the number of linearly independent rows or columns. The rank can be found by finding the highest-order square submatrix that is nonsingular. The determinant of $\mathbf{O}_M = -344$. Since the determinant is not zero, the 3X3 matrix is nonsingular, and the rank of $\mathbf{C}_M$ is of full rank equal to 3.

A=[0 1 0;0 0 1;-4 -3 -2]

C=[0 5 1]

Om=obsv(A,C)

Rank=rank(Om)

% Define compensated A matrix.

% Define compensated C matrix.

% Calculate observability matrix.

% Find rank of observability matrix.

Controllability and Observability

$$A = \begin{bmatrix} 0 & 6 & -5 \\ 1 & 0 & 2 \\ 3 & 2 & 4 \end{bmatrix} \text{ --- (1)}$$

$$B = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \text{ --- (2)}$$

$$C = [1 \quad 2 \quad 3] \text{ --- (3)}$$

Since A is a 3x3 matrix, so n=3.

Controllability: The necessary and sufficient condition for controllability is

$$Q_C = [B \quad AB \quad \dots \quad A^{n-1}B]$$

$$Q_C = [B \quad AB \quad A^2B] \text{ --- (4)}$$

From equation (2),

$$B = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \text{ --- (5)}$$

$$A \cdot B = \begin{bmatrix} 0 & 6 & -5 \\ 1 & 0 & 2 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} -4 \\ 4 \\ 10 \end{bmatrix} \text{ --- (6)}$$

$$A^2B = A \cdot (AB) = \begin{bmatrix} 0 & 6 & -5 \\ 1 & 0 & 2 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} -4 \\ 4 \\ 10 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} -26 \\ 16 \\ 36 \end{bmatrix} \text{ --- (7)}$$

Put equation (5), (6), (7) in equation (4),

$$Q_C = \begin{bmatrix} 0 & -4 & -26 \\ 1 & 4 & 16 \\ 2 & 10 & 36 \end{bmatrix}$$

Now, find the determinant of Q_C

$$|Q_C| = \begin{vmatrix} 0 & -4 & -26 \\ 1 & 4 & 16 \\ 2 & 10 & 36 \end{vmatrix}$$

$$|Q_C| = -36$$

$$\therefore |Q_C| \neq 0$$

Since the determinant of Q_C is non-zero, therefore of $Q_C = n = 3$. Hence, the system is completely controllable.

Controllability and Observability



Observability:

Given,

$$B = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$$B' = [0 \quad 1 \quad 2]$$

$$C = [1 \quad 2 \quad 3]$$

$$C' = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 6 & -5 \\ 1 & 0 & 2 \\ 3 & 2 & 4 \end{bmatrix}$$

$$A' = \begin{bmatrix} 0 & 1 & 3 \\ 6 & 0 & 2 \\ -5 & 2 & 4 \end{bmatrix}$$

The necessary and sufficient condition for observability.

$$Q_C = [C^T \quad A^T C^T \quad \dots \quad (A^T)^{n-1} C^T]$$

For n=3,

$$Q_C = [C^T \quad A^T C^T \quad (A^T)^2 C^T] \quad \text{--- (8)}$$

$$C^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{--- (9)}$$

$$A^T C^T = \begin{bmatrix} 0 & 1 & 3 \\ 6 & 0 & 2 \\ -5 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$A^T C^T = \begin{bmatrix} 11 \\ 12 \\ 11 \end{bmatrix} \quad \text{--- (10)}$$

$$(A^T)^2 C^T = A^T \cdot (A^T C^T) = \begin{bmatrix} 0 & 1 & 3 \\ 6 & 0 & 2 \\ -5 & 2 & 4 \end{bmatrix} \begin{bmatrix} 11 \\ 12 \\ 11 \end{bmatrix}$$

$$(A^T)^2 C^T = \begin{bmatrix} 45 \\ 88 \\ 13 \end{bmatrix} \quad \text{--- (11)}$$

Put equations (9), (10), (11) in equation (8),

$$Q_C = \begin{bmatrix} 1 & 11 & 45 \\ 2 & 12 & 88 \\ 3 & 11 & 13 \end{bmatrix}$$

Now, find the determinant of Q_C

$$|Q_C| = \begin{vmatrix} 1 & 11 & 45 \\ 2 & 12 & 88 \\ 3 & 11 & 13 \end{vmatrix}$$

$$|Q_C| = 1176$$

Since, $|Q_C| \neq 0$, the rank of Q_C is n=3.

∴ The system is completely observable.

State Space Representations

Example 1: Conversion from Transfer-Function Model to State-Space Model

$$\frac{Y(s)}{U(s)} = \frac{s}{s^3 + 14s^2 + 56s + 160}$$

MATLAB code:

```
n=[1 0]; d=[1 14 56 160];
```

```
[Ac,Bc,Cc,Dc]=tf2ss(n,d) %Controller canonical form
```

```
system=tf(n,d)
```

```
[a,b,c,d]=ssdata(system) % Not controller canonical form
```

State Space Representations

Example 2:

s

$$s^3 + 14s^2 + 56s + 160$$

$A_c =$

$$\begin{bmatrix} -14 & -56 & -160 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$B_c =$

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$C_c =$

$$\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$$

$D_c =$

$$\begin{bmatrix} 0 \end{bmatrix}$$

$a =$

$$\begin{bmatrix} -14 & -7 & -5 \\ 8 & 0 & 0 \\ 0 & 4 & 0 \end{bmatrix}$$

$b =$

$$\begin{bmatrix} 0.2500 \\ 0 \\ 0 \end{bmatrix}$$

$c =$

$$\begin{bmatrix} 0 & 0.5000 & 0 \end{bmatrix}$$

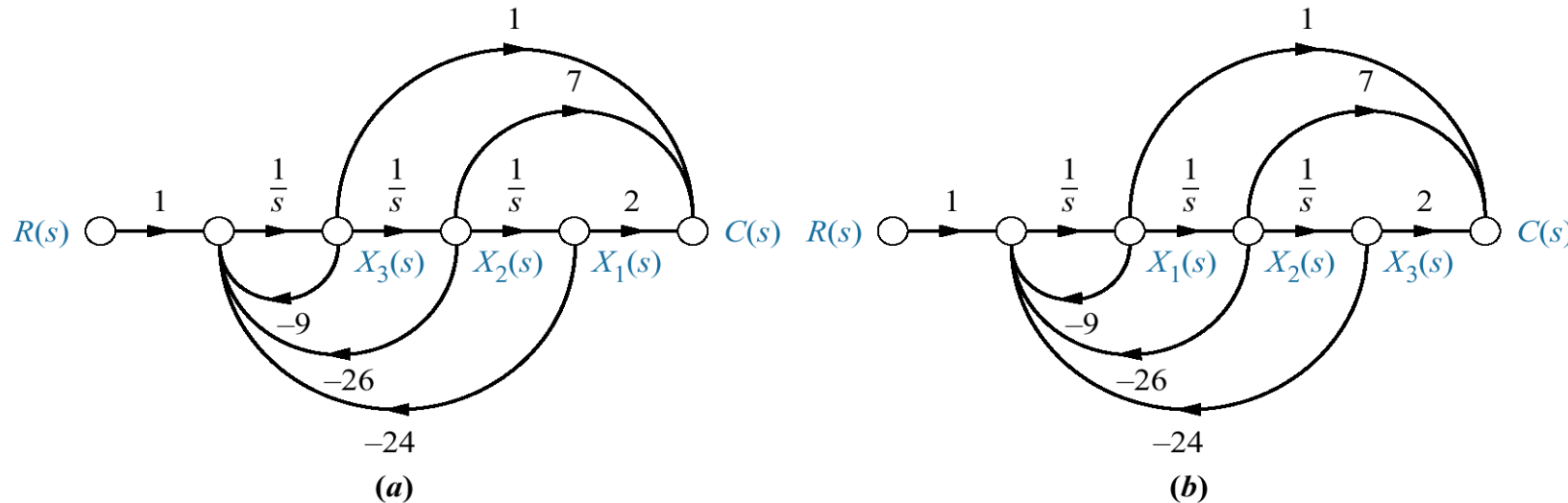
$d =$

$$\begin{bmatrix} 0 \end{bmatrix}$$

Similarity between phase-variable & controller canonical forms

Signal-flow graphs for obtaining forms for

$$G(s) = C(s)/R(s) = (s^2 + 7s + 2)/(s^3 + 9s^2 + 26s + 24)$$



a. phase-variable form;

b. controller canonical form

Forming the Controllability Matrix

Example3:

For the canonical state-space model defined in Example 2

$$\mathbf{CO} = \text{ctrb}(\mathbf{Ac}, \mathbf{Bc})$$

$$\mathbf{CO} = \begin{bmatrix} 1 & -14 & 140 \\ 0 & 1 & -14 \\ 0 & 0 & 1 \end{bmatrix}$$

Now we can check its determinant to make sure its inverse exists:

$$\det(\mathbf{CO})$$

$$\text{ans} = 1$$

Therefore, the system is controllable.

An Uncontrollable System

Example 4: Let $\mathbf{a}=[1 \ 1;0 \ 1]$; $\mathbf{b}=[1;0]$; $\mathbf{CO}=\text{ctrb}(\mathbf{a},\mathbf{b})$; $\det(\mathbf{CO})$

ans = 0

Since the controllability matrix is singular the system is uncontrollable. Note the transfer function model has pole-zero cancellation at $s = -1$. This is symptomatic of an uncontrollable mode.

$[\text{zeros},\text{poles}]=\text{ss2zp}(\mathbf{a},\mathbf{b},[1 \ 1],0)$

zeros =
1.0000

poles =
1
1

END