

MENG366

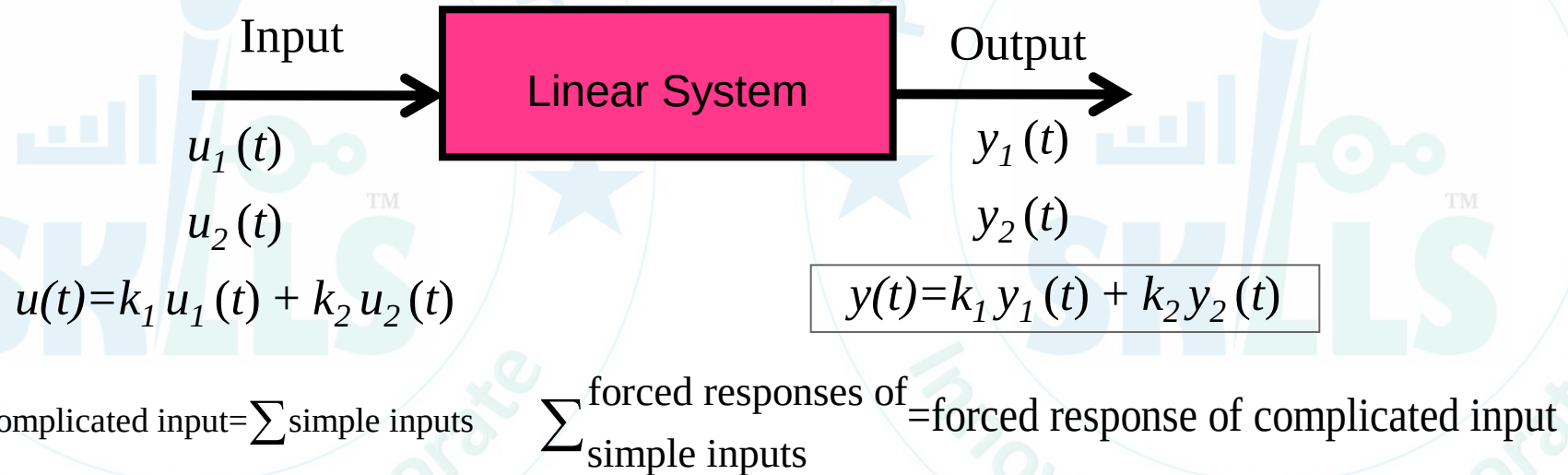
Transient and Steady State Response of First, Second, and Higher Order Systems

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Forced Responses of LTI Systems

Superposition Principle

$$y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_2\ddot{y} + a_1\dot{y} + a_0y = b_mu^{(m)} + \dots + b_1\dot{u} + b_0u$$



The forced response of a linear system to a complicated input can be obtained by studying how the system responds to simple inputs, such as **unit impulse input**, **unit step input**, and **sinusoidal inputs** with different input frequencies.

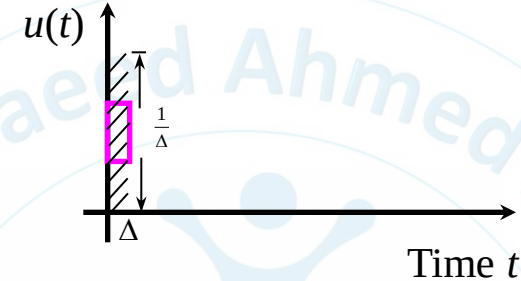
Typical Forced Responses

• Unit Impulse Response

- Forced response to unit impulse input

$$u(t) = \delta(t) = \begin{cases} +\infty, & t = 0 \\ 0, & t \neq 0 \end{cases}, \quad \int_{-\infty}^{+\infty} \delta(t) dt = 1 \quad Y(s) = G(s)U(s) = G(s)$$

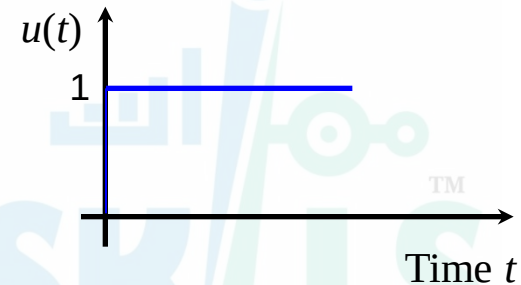
If system is stable, SS is zero.



• Unit Step Response

- Forced response to unit step input ($u(t) = 1$)

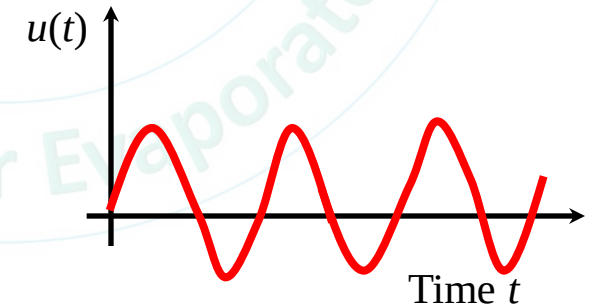
$$Y(s) = G(s)U(s) = \frac{1}{s} G(s) \quad y(\infty) = \lim_{s \rightarrow 0} sY(s) = \lim_{s \rightarrow 0} s \frac{1}{s} G(s) = G(0)$$



• Sinusoidal Response

- Forced response to sinusoidal inputs at different input frequencies
- The *steady state response* of sinusoidal response is: *Frequency Response*.

$$Y(s) = G(s)U(s) = G(s) \underbrace{\frac{\omega}{s^2 + \omega^2}}_{U(s)} \quad sY(s) = \frac{s\omega}{s^2 + \omega^2} G(s)$$



Forced Response of 1st Order Systems

- Standard Form of *Stable* 1st Order System

$$\dot{y} + ay = bu \Rightarrow \tau \dot{y} + y = Ku$$

$$\frac{1}{a} \dot{y} + y = \frac{b}{a} u$$

where

τ : Time Constant
 K : Static (Steady State, DC) Gain

$$\tau = \frac{1}{a}$$

$$K = \frac{b}{a}$$

- TF and Poles/Zeros

$$G(s) = \frac{K}{\tau s + 1}$$

pole: $p = -\frac{1}{\tau} < 0$

zero: No zero

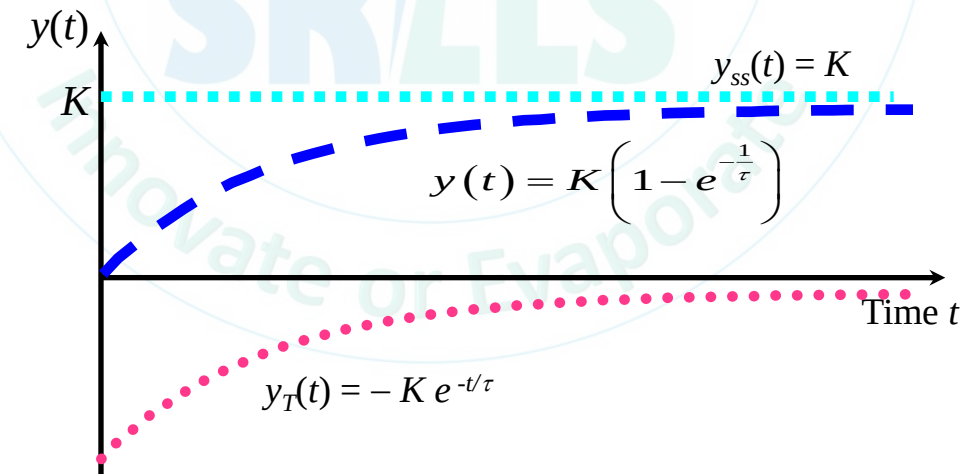
Stable system

- Unit Step Response

– ($u=1$ and zero ICs)

$$Y(s) = G(s)U(s) = \frac{K}{(\tau s + 1)s} = K \left[\frac{1}{s} - \frac{\tau}{\tau s + 1} \right] = K \left[\frac{1}{s} - \frac{1}{s + \frac{1}{\tau}} \right]$$

$$y(t) = K \left(1 - e^{-\frac{t}{\tau}} \right)$$



Normalized Unit Step Response

Normalized Unit Step Response ($u = 1$ & zero ICs)

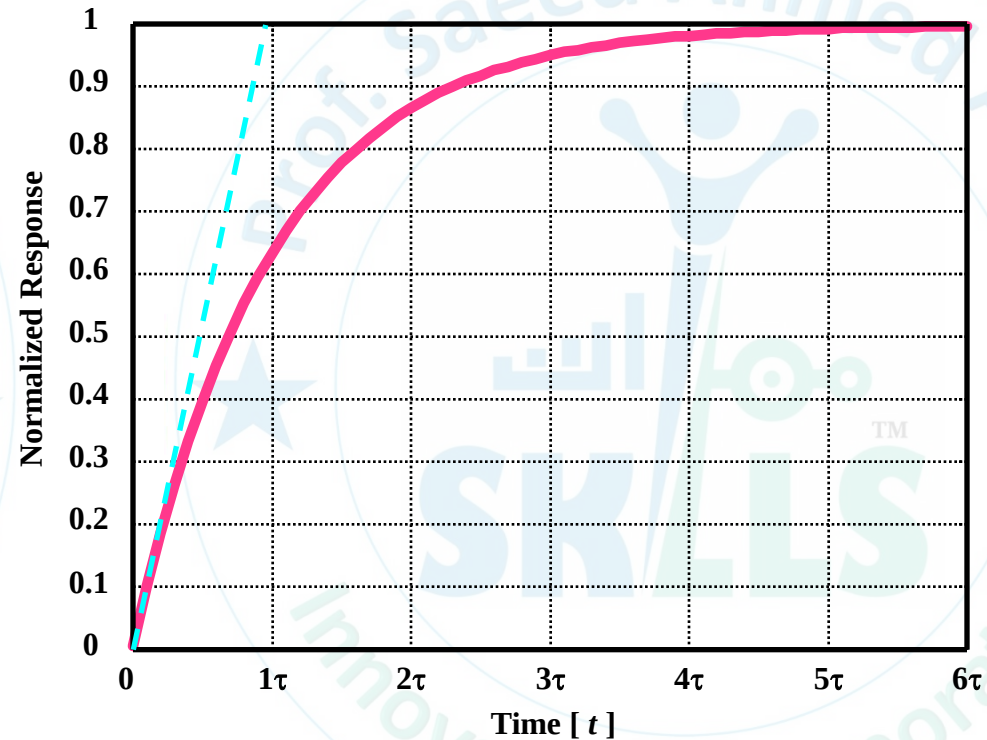
$$\tau \dot{y} + y = Ku$$

$$\Rightarrow y(t) = K(1 - e^{-t/\tau})$$

Normalized

(such that as $t \rightarrow \infty, y_n \rightarrow 1$):

$$\Rightarrow y_n(t) = \frac{y(t)}{K} = 1 - e^{-t/\tau}$$



Time t	τ	2τ	3τ	4τ	5τ
$(1 - e^{-t/\tau})$	0.6321	0.8647	0.9502	0.9817	0.9933

Unit Step Response of Stable 1st Order System

Effect of Time Constant τ :

$$\tau \dot{y} + y = Ku$$

$$\Rightarrow y(t) = K(1 - e^{-t/\tau})$$

Normalized:

$$\Rightarrow y_n(t) = \frac{y(t)}{K} = (1 - e^{-t/\tau})$$

Initial Slope:

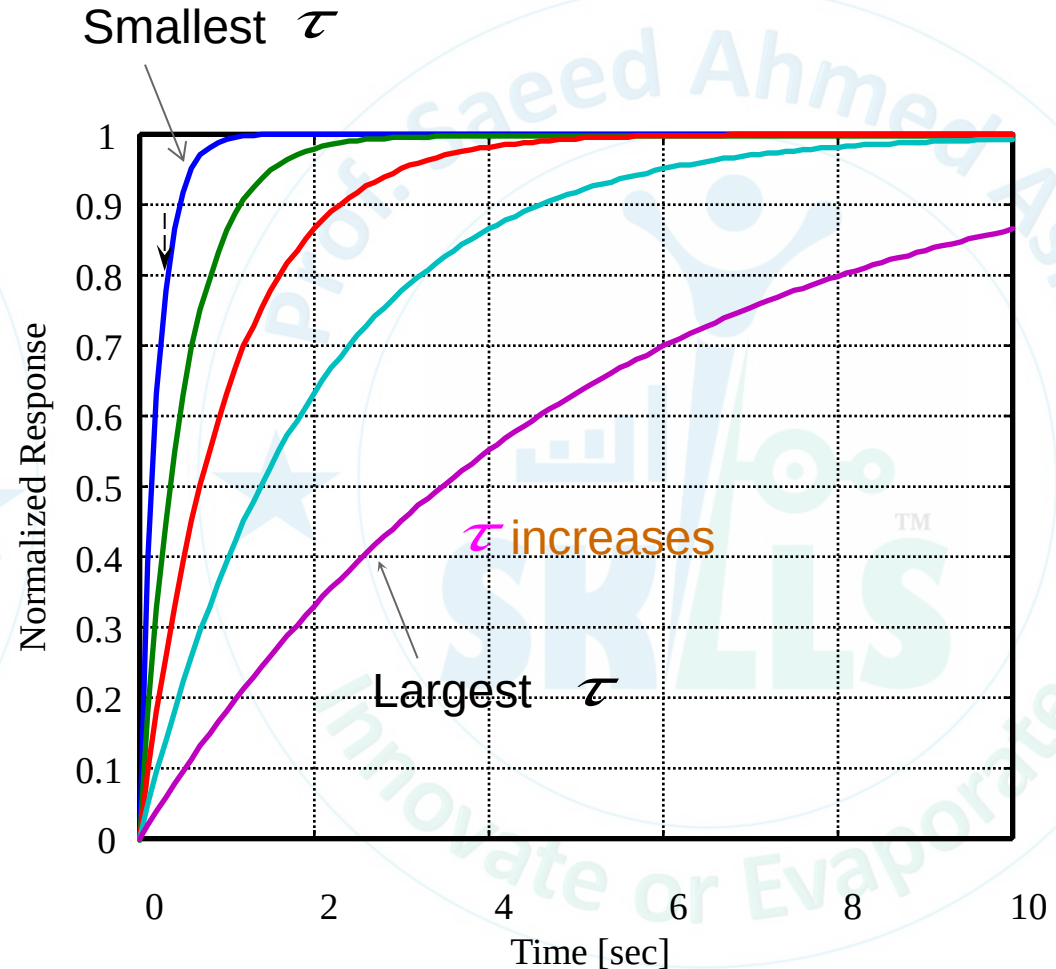
$$\Rightarrow \frac{d}{dt} y_n(t) = \frac{1}{\tau} e^{-t/\tau}$$

$$\Rightarrow \frac{d}{dt} y_n(0) = \frac{1}{\tau}$$

Q: What is your conclusion ?

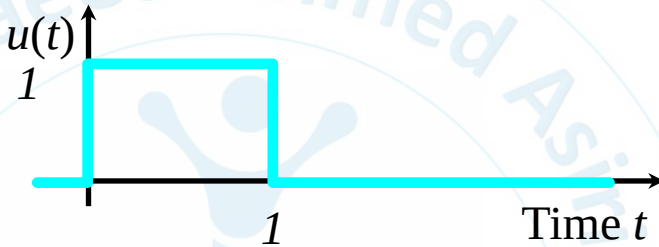
The smaller τ is,

the steeper the initial slope is, and the faster the response approaches the steady state.

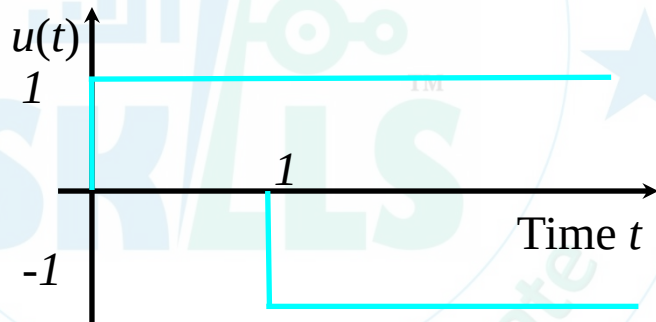


Forced Responses of Stable 1st Order System

Q: How would you calculate the forced response of a 1st order system to a unit pulse (not unit impulse)?



Q (Hint: superposition principle ?!)



$$u(t) = u_s(t) - u_s(t-1)$$

$$y(t) = y_s(t) - y_s(t-1)$$

Q: How would you calculate the unit impulse response of a 1st order system?

$$\delta(t) = \frac{d}{dt} u_s(t)$$

$$y(t) = \frac{d}{dt} y_s(t)$$

Q: How would you calculate the sinusoidal response of a 1st order system?

Standard Form of 2nd Order Systems

- I/O Model

$$\ddot{y} + a_1 \dot{y} + a_0 y = b_1 \dot{u} + b_0 u$$

- TF and Pole/Zeros

$$G(s) = \frac{b_1 s + b_0}{s^2 + a_1 s + a_0} \quad \text{pole: } p_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0}}{2} \quad \text{zero: } z = -\frac{b_0}{b_1} \quad \omega_n = \sqrt{a_0}$$

- Stability Condition

$$a_1 > 0, \quad a_0 > 0$$

- Standard Form of *Stable* 2nd Order Systems without Zeros

$$\ddot{y} + a_1 \dot{y} + a_0 y = b_0 u \Rightarrow \ddot{y} + 2\zeta\omega_n \dot{y} + \omega_n^2 y = K\omega_n^2 u$$

where

ω_n : Natural Frequency [rad/s]

ζ : Damping Ratio $\zeta = \frac{a_1}{2\omega_n} = \frac{a_1}{2\sqrt{a_0}}$

K : Static (Steady State, DC) Gain $K = \frac{b_0}{\omega_n^2} = \frac{b_0}{a_0}$

Poles of Stable 2nd Order Systems

Stable 2nd Order Systems without Zeros

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = K\omega_n^2 u$$

$$G(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{K}{\frac{1}{\omega_n^2}s^2 + \frac{2\zeta}{\omega_n}s + 1}$$

Pole Locations

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

$$\Rightarrow p_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{(\zeta^2 - 1)}$$

Over-damped ($\zeta > 1$)

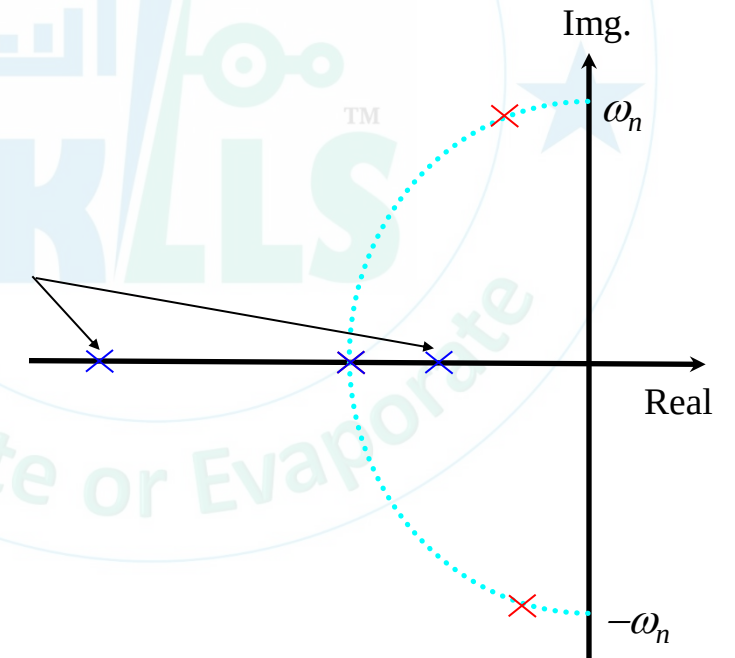
Two distinct real poles

Critically damped ($\zeta = 1$)

Two identical real poles at $p_{1,2} = -\zeta\omega_n$

Under damped ($\zeta < 1$)

Two complex poles at $p_{1,2} = -\underbrace{\zeta\omega_n}_{\sigma} \pm j\underbrace{\omega_n\sqrt{1-\zeta^2}}_{\omega_d}$

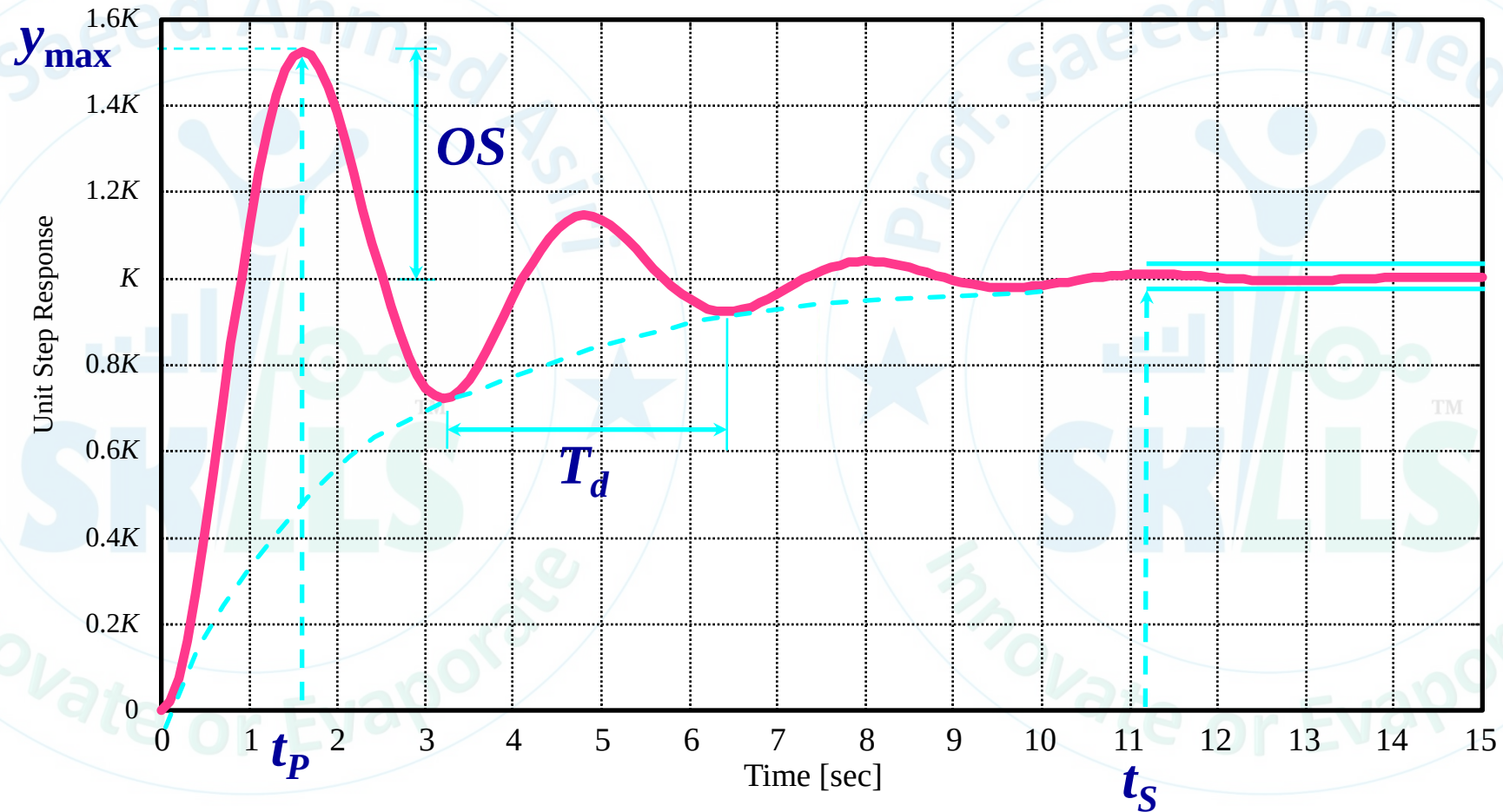


Under-damped 2nd Order System

- Unit Step Response ($u=1$ and zero ICs)

$$\begin{aligned}
 Y(s) &= G(s) \frac{1}{s} = \frac{K\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} = \frac{K\omega_n^2}{s(s - p_1)(s - p_2)}, \quad p_{1,2} = -\sigma \pm j\omega_d \\
 &= \frac{K}{s} + \frac{A_1}{s + \sigma - j\omega_d} + \frac{\bar{A}_1}{s + \sigma + j\omega_d}, \quad A_1 = -\frac{K}{2} \left(1 - j \frac{\sigma}{\omega_d} \right) \\
 \Rightarrow y(t) &= K + \underbrace{A_1 e^{(-\sigma + j\omega_d)t} + \bar{A}_1 e^{(-\sigma - j\omega_d)t}}_{2 \operatorname{Re}\{A_1 e^{(-\sigma + j\omega_d)t}\}} \\
 &= K \left[1 - e^{-\sigma t} \left(\cos(\omega_d t) + \frac{\zeta\omega_n}{\omega_d} \sin(\omega_d t) \right) \right] \\
 &= K - \frac{K}{\sqrt{1 - \zeta^2}} e^{-\sigma t} \sin(\omega_d t + \varphi), \quad \varphi = \tan^{-1} \left(\frac{\sqrt{1 - \zeta^2}}{\zeta} \right)
 \end{aligned}$$

Unit Step Response of 2nd Order Systems



Unit Step Response of 2nd Order System

• Peak Time (t_p)

Time when output $y(t)$ reaches its maximum value y_{MAX} .

$$y(t) = K \left[1 - e^{-\sigma t} \left(\cos(\omega_d t) + \frac{\zeta \omega_n}{\omega_d} \sin(\omega_d t) \right) \right]$$

$\Rightarrow$

$$\frac{dy}{dt} = K e^{-\zeta \omega_n t} \frac{\omega_n}{\sqrt{1-\zeta^2}} \sin(\omega_d t)$$

Find t_p such that $\dot{y}(t_p) = 0$

$$\omega_d t_p = \pi$$

$\Rightarrow$

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}}$$

$$T_d = \frac{2\pi}{\omega_n \sqrt{1-\zeta^2}} = \frac{2\pi}{\omega_d}$$

• Percent Overshoot (%OS)

At peak time t_p the maximum output

$$y_{MAX} = y(t_p) = K \left(1 + e^{-\frac{\pi \zeta}{\sqrt{1-\zeta^2}}} \right)$$

The overshoot (OS) is:

$$OS = y_{MAX} - y_{SS} = K e^{-\frac{\pi \zeta}{\sqrt{1-\zeta^2}}}$$

The percent overshoot is:

$$\%OS = \left(\frac{OS}{y_{SS} - y(0)} \right) 100\%$$

$$\%OS = e^{-\frac{\pi \zeta}{\sqrt{1-\zeta^2}}} \times 100\%$$

Unit Step Response of 2nd Order System

• Settling Time (t_s)

Time required for the response to be within a specific percent of the final (steady-state) value.

Some typical specifications for settling time are: 5%, 2% and 1%.

Look at the envelope of the response:

x% band settling time:

$$t_s = -\tau \ln\left(\frac{x}{100}\right) \approx$$

$$\tau = 1/\sigma$$

%	1%	2%	5%
t_s	4.6τ	3.9τ	3τ

Q: Which parameters of a 2nd order system affect the peak time?

Damping ratio and natural frequency

Q: Which parameters of a 2nd order system affect the % OS?

Damping ratio

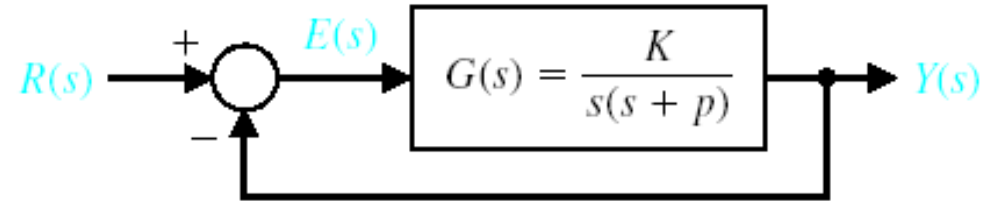
Q: Which parameters of a 2nd order system affect the settling time?

Damping ratio and natural frequency

Q: Can you obtain the formula for a 3% settling time?

Performance of a Second-Order System

$$Y(s) = \frac{K}{s^2 + p \cdot s + K} \cdot R(s)$$



$$Y(s) = \frac{\omega_n^2}{s^2 + 2 \cdot \zeta \cdot \omega_n \cdot s + \omega_n^2}$$

with a unity step input

$$Y(s) = \frac{\omega_n^2}{(s^2 + 2 \cdot \zeta \cdot \omega_n \cdot s + \omega_n^2) \cdot s}$$

$$y(t) = 1 - \frac{1}{\beta} \cdot e^{-\zeta \cdot \omega_n \cdot t} \cdot \sin(\omega_n \cdot \beta \cdot t + \theta)$$

$$\beta = \sqrt{1 - \zeta^2}$$

$$\theta = \cos^{-1}(\zeta)$$

Performance of a Second-Order System

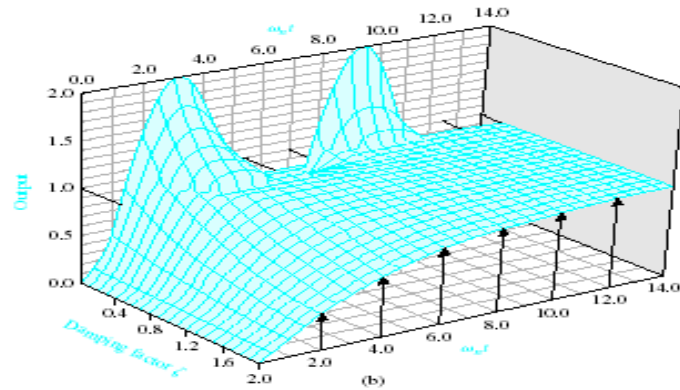
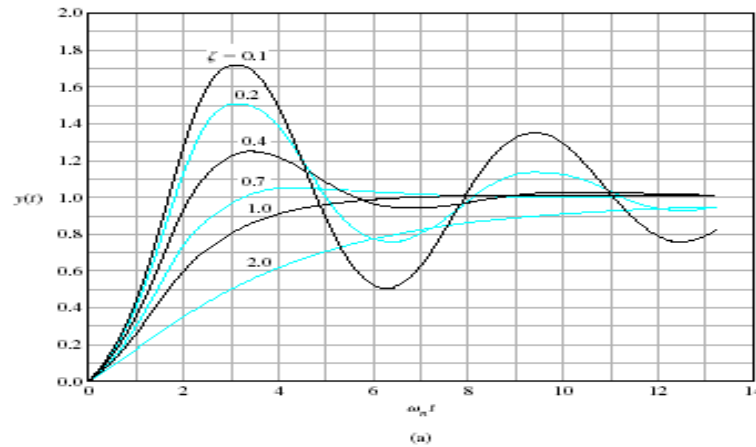
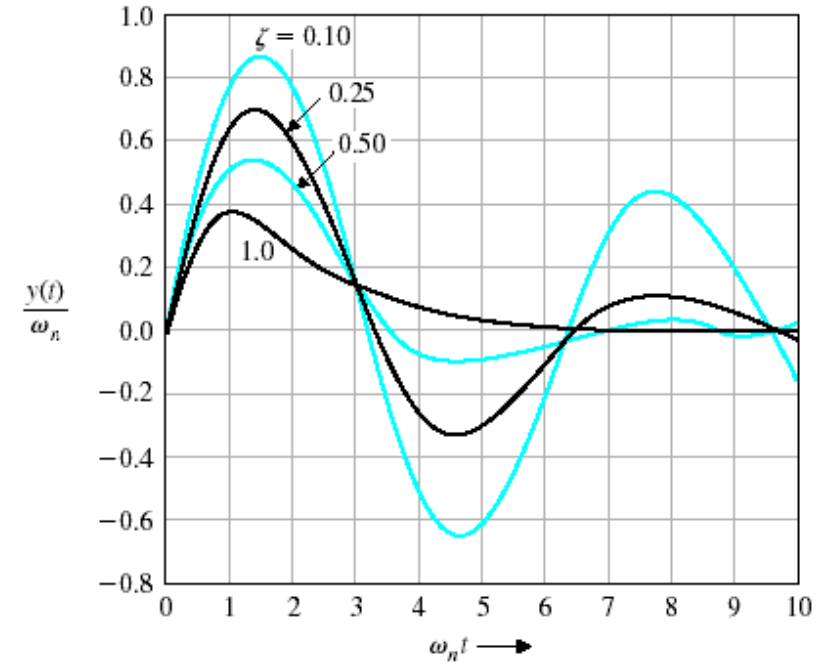
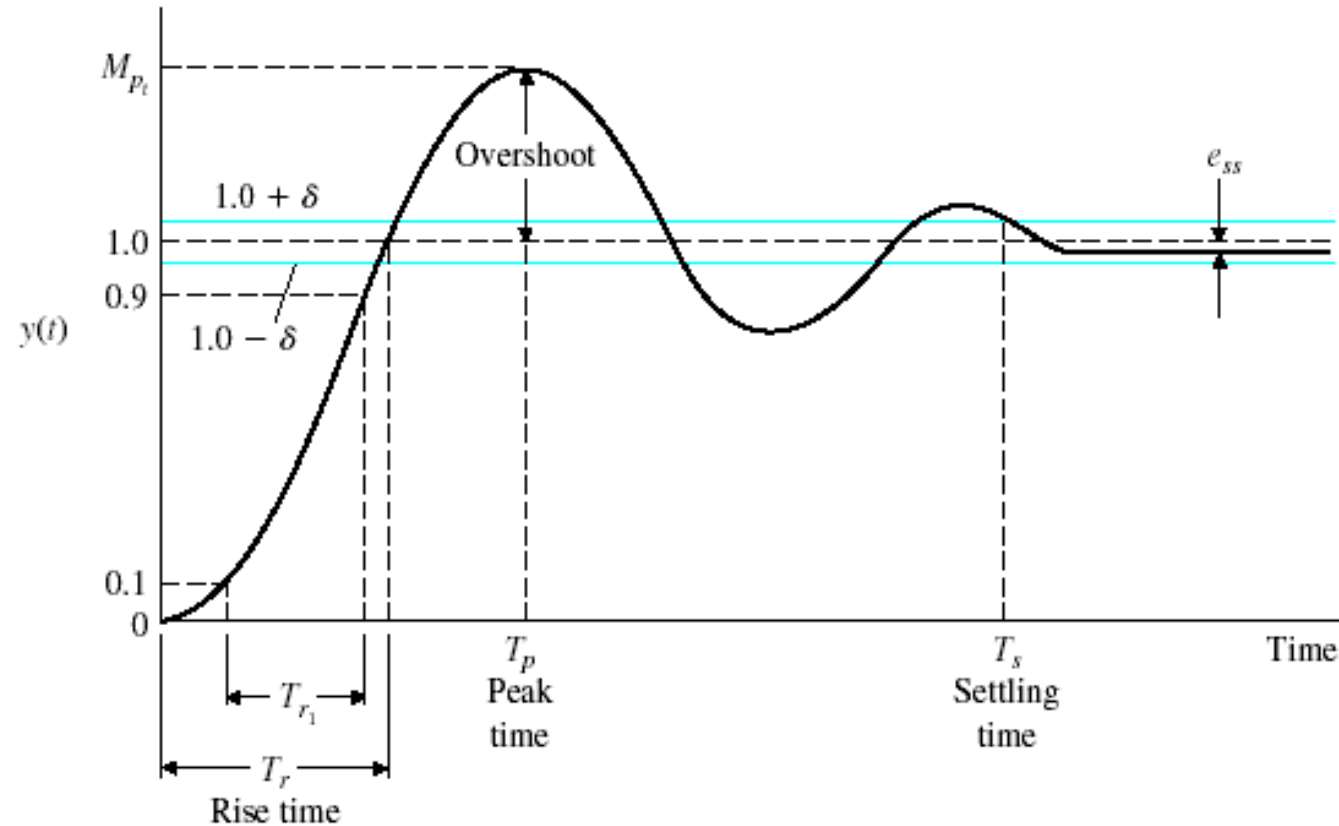


FIGURE 5.5

(a) Transient response of a second-order system (Eq. 5.9) for a step input. (b) The transient response of a second-order system (Eq. 5.9) for a step input as a function of ζ and $\omega_n t$. (Courtesy of Professor R. Jacquot, University of Wyoming.)



Response of a second-order system for an impulse function input.



Step response of a control system (Eq. 5.9).

Rise Time, T_r
 Peak Time, T_p
 Percentage Overshoot, P.O.
 Settling Time, T_s
 Normalized Rise Time T_{r1}

Performance of a Second-Order System

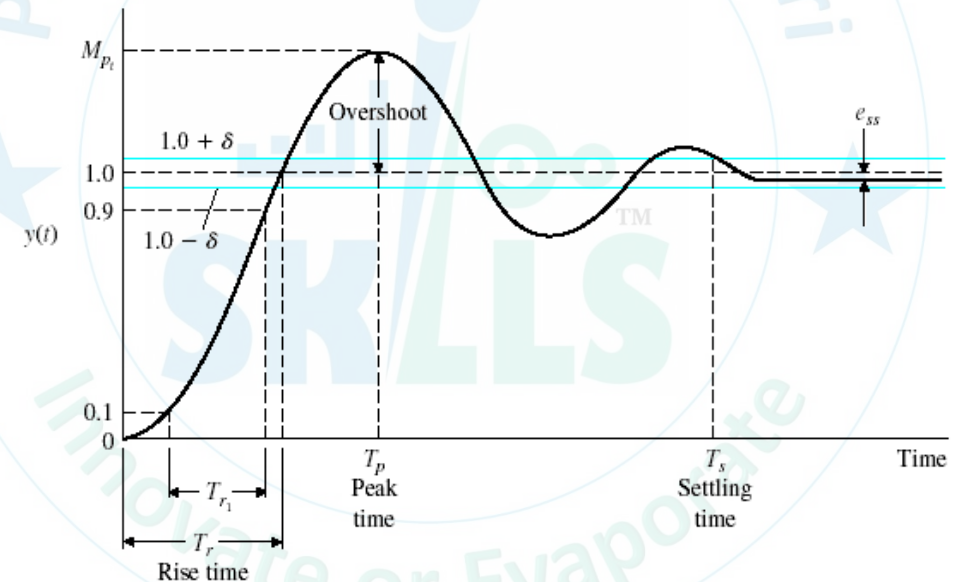
Standard performance measures are usually defined in terms of the step response of a system. The transient response of a system may be described using two factors, the swiftness and the closeness of the response to the desired response.

The swiftness of the response is measured by the rise time (T_r) and the peak time (T_p).

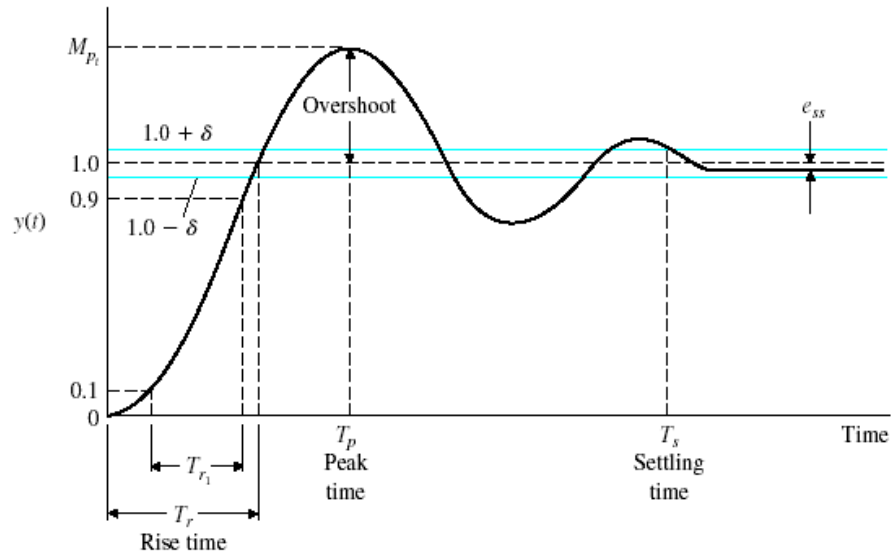
Underdamped systems: 0-100% rise time is used

Overdamped systems: 10-90% rise time is used

The closeness is measured by the overshoot and settling time. Using these measurements, the percent overshoot (P.O.) can be calculated.



Performance of a Second-Order System



$$PO = \frac{M_{pv} - f_v}{f_v} \cdot 100$$

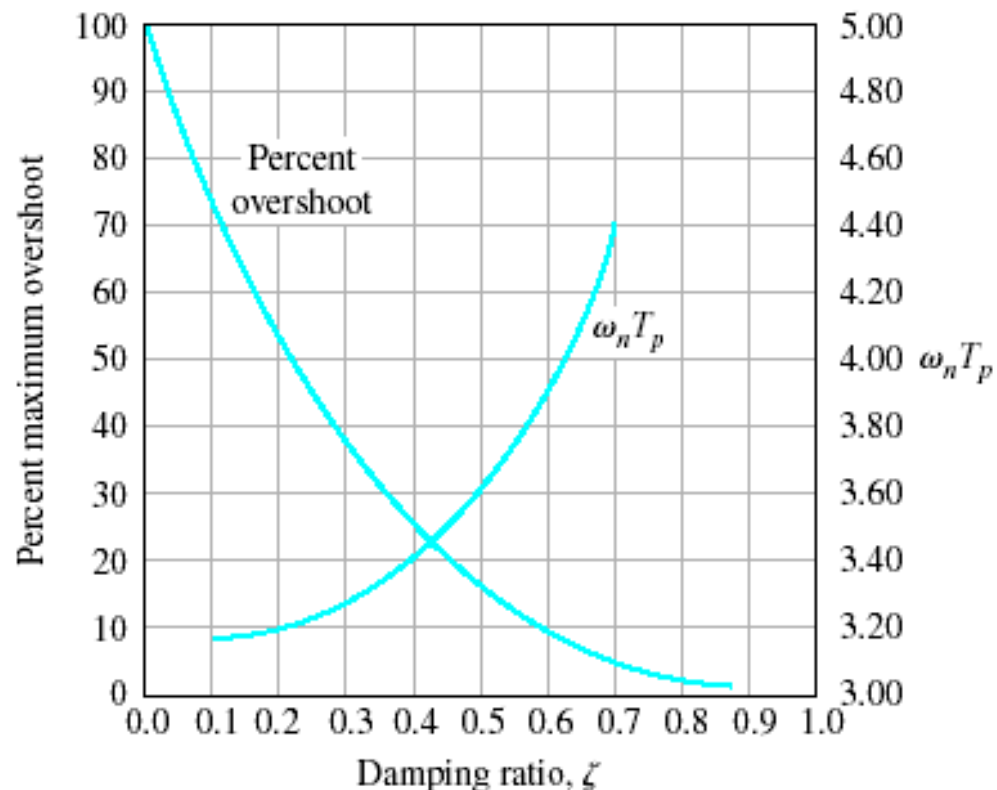
$$T_s = \frac{4}{\zeta \cdot \omega_n}$$

$$T_p = \frac{\pi}{\omega_n \cdot \sqrt{1 - \zeta^2}}$$

$$M_{pv} = 1 + e^{-\zeta \cdot \frac{\pi}{\sqrt{1 - \zeta^2}}}$$

$$PO = 100 \cdot e^{-\zeta \cdot \frac{\pi}{\sqrt{1 - \zeta^2}}}$$

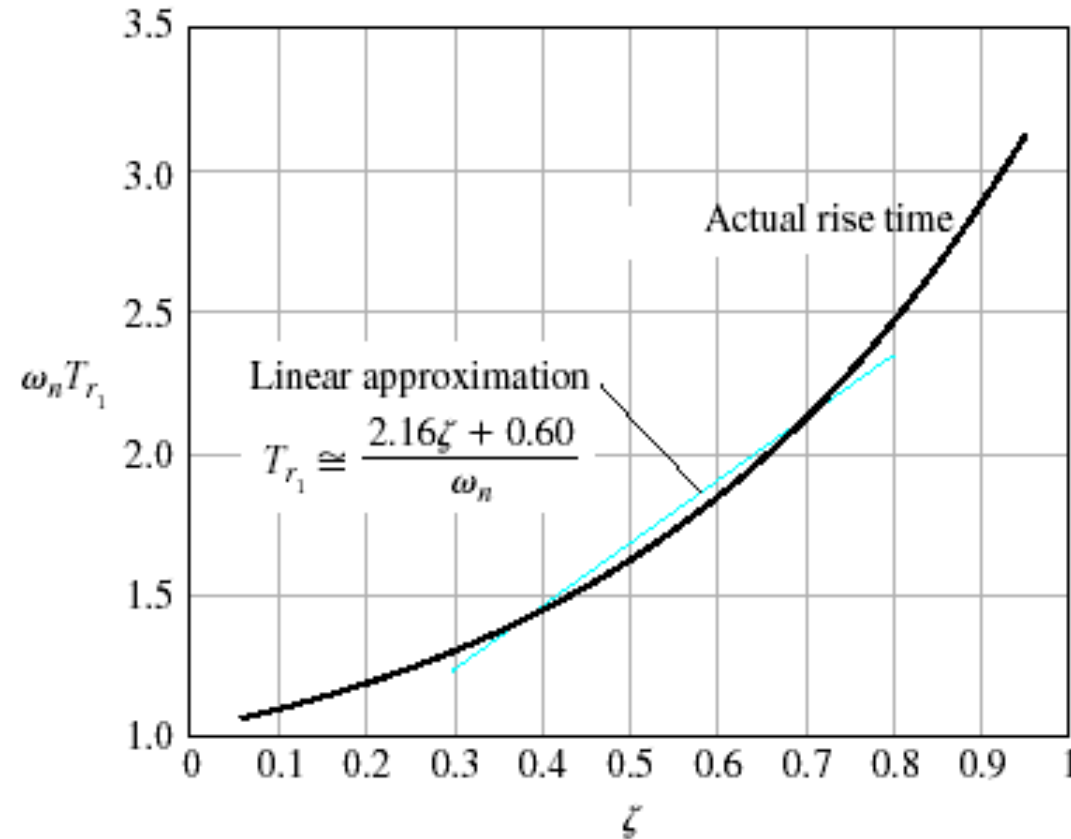
Performance of a Second-Order System



Percent overshoot and normalized peak time versus damping ratio ζ for a second-order system (Eq. 5.8).

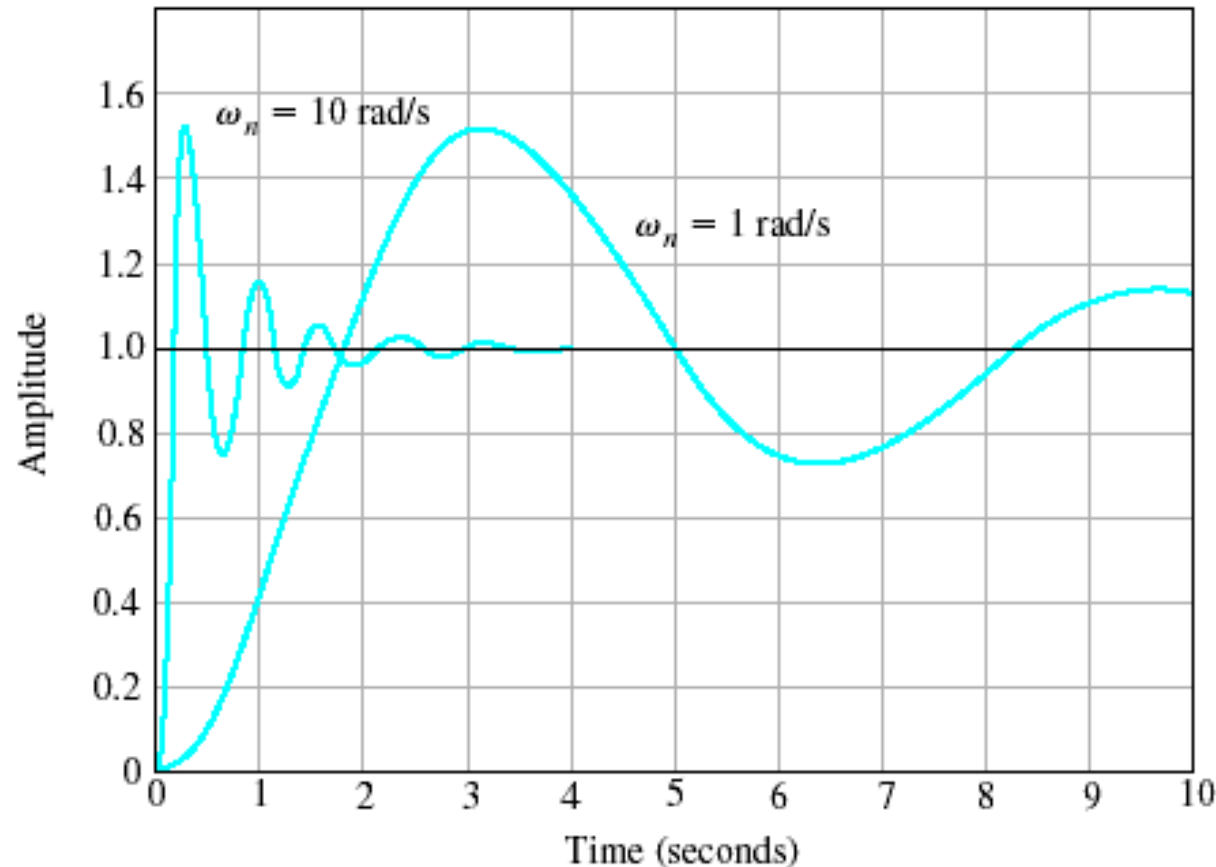
Naturally these two performance measures are in opposition and a compromise must be made.

Performance of a Second-Order System



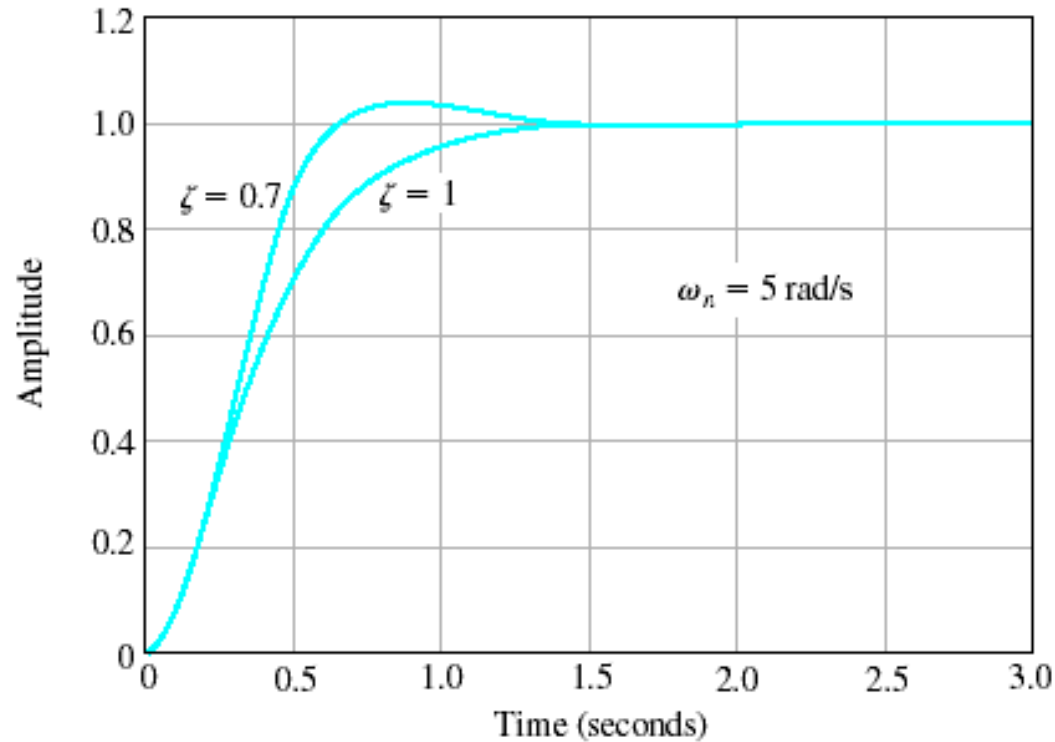
Normalized rise time T_{r1} versus ζ for a second-order system.

Performance of a Second-Order System



The step response for $\zeta = 0.2$ for $\omega_n = 1$ and $\omega_n = 10$.

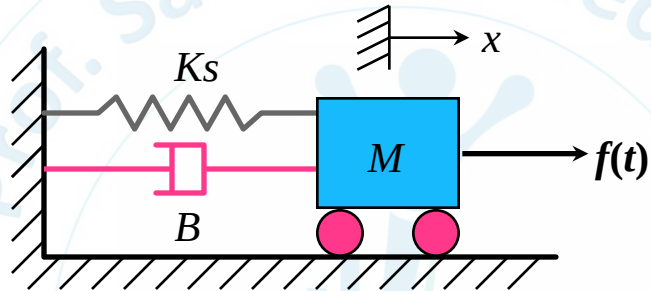
Performance of a Second-Order System



The step response for $\omega_n = 5$ with $\zeta = 0.7$ and $\zeta = 1$.

Performance of a Second-Order System

- Mass-Spring-Damper System**



I/O Model:

$$M \ddot{x} + B \dot{x} + K_s x = f(t)$$

$$G(s) = \frac{1}{Ms^2 + Bs + K_s}$$

$$= \frac{1/M}{s^2 + \frac{B}{M}s + \frac{K_s}{M}}$$

$$= \frac{K \omega_n^2}{s^2 + 2\zeta \omega_n s + \omega_n^2}$$

$$\omega_n = \sqrt{\frac{K_s}{M}}$$

$$\zeta = \frac{B}{2\sqrt{MK_s}}$$

$$K = \frac{1}{K_s}$$

Q: What is the static gain of the system ?

Q: How would the physical parameters (M , B , K) affect the step response of the system ?

(This is equivalent to asking you for the relationship between the physical parameters and the damping ratio, natural frequency and the static gain.)